# Quasi-equilibrium binary black hole initial data

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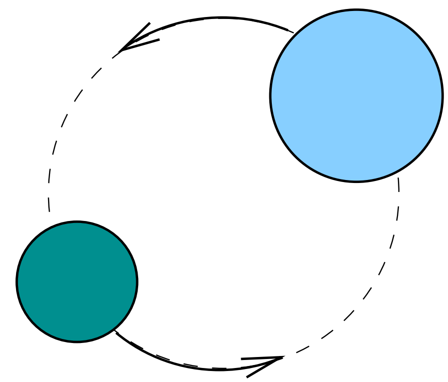
**Outline:**

1. Formalism & Numerics
2. Non-uniqueness in conformal thin sandwich
3. Properties of the constructed ID sets
4. Public initial data repository

# Formalism & Numerics

# Quasi-equilibrium method

**Basic idea:**
Approx. time-independence in corotating frame
Approx. helical Killing vector
(both concepts essentially equivalent,
both useful depending on context)



**History:**

- Wilson & Matthews 1985: Binary neutron stars
- Gourgoulhon, Grandclement & Bonazzola, 2002a,b
  BBH ID with inner boundary conditions
  basically right, but various deficiencies
- Cook & HP, 2002, 2003, 2004 (especially Cook & Pfeiffer, PRD 70, 104106, 2004)
  General quasi-equilibrium method with isolated horizon BCs

## Quasi-equilibrium method (the easy pieces)

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$$\tilde{\nabla}^2\psi - \frac{1}{8}\tilde{R}\psi - \frac{1}{12}K^2\psi^4 + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij} = 0$$

$$\tilde{\nabla}_j\left(\frac{\psi^6}{2N}\mathbb{L}\beta^{ij}\right) - \frac{2}{3}\psi^6\tilde{\nabla}^i K - \tilde{\nabla}_j\left(\frac{\psi^6}{2N}\tilde{u}^{ij}\right) = 0$$

$$\tilde{\nabla}^2(N\psi) - N\psi\left(\frac{1}{8}\tilde{R} + \frac{5}{12}K^2\psi^4 + \frac{7}{8}\tilde{A}_{ij}\tilde{A}^{ij}\right) = -\psi^5(\partial_t - \beta^k\partial_k)K$$

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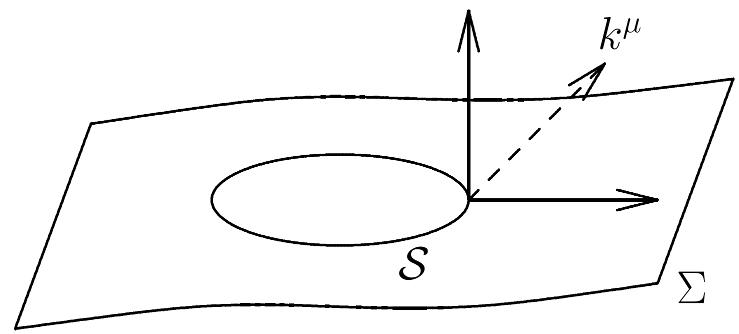
- **Boundary conditions at infinity**

  $$\psi = 1$$
  $$\beta^i = (\vec{\Omega}_{\text{orbital}} \times \vec{r})^i$$
  $$N = 1$$

- **New contribution**: *inner boundary conditions* (next slides)

# Quasi-equilibrium excision boundary conditions

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- **Require**
  1. $\mathcal{S}$ be apparent horizons
  2. The AH's remain stationary in evolution
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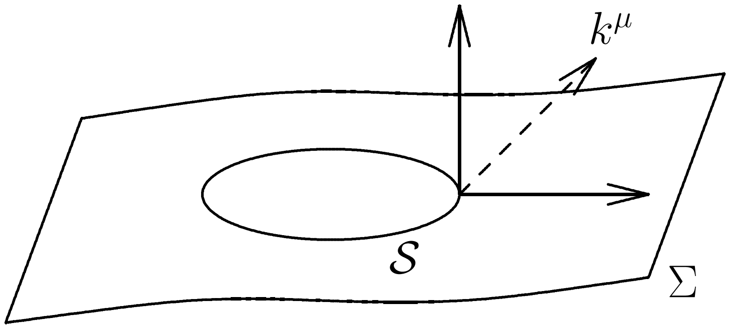


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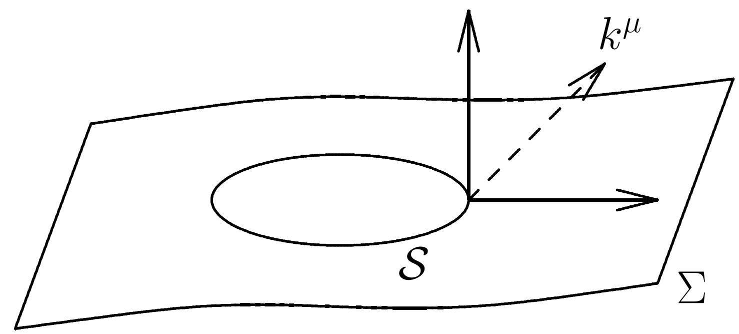


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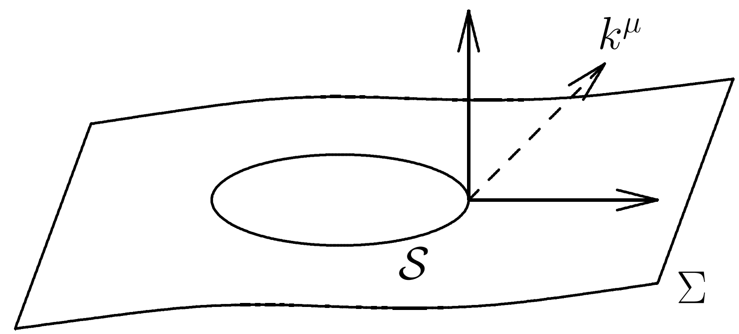


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- **Rewrite** in conformal variables $\Rightarrow$ BC's on $\psi$ and $\beta^i$
- General spin possible ($\longrightarrow$ Greg Cook's talk)
- One still must specify...
  1. Conformal metric $\tilde{g}_{ij}$
  2. Shape of excision surfaces $\mathcal{S}$
  3. Mean curvature $K$
  4. Lapse boundary condition

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Expand solution in basis-functions & solve for expansion-coefficients

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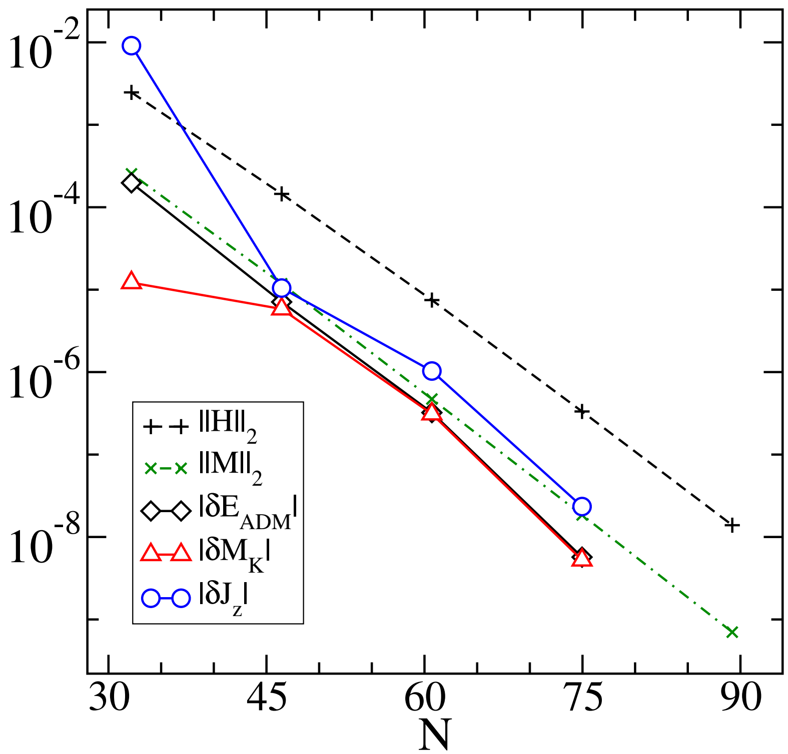

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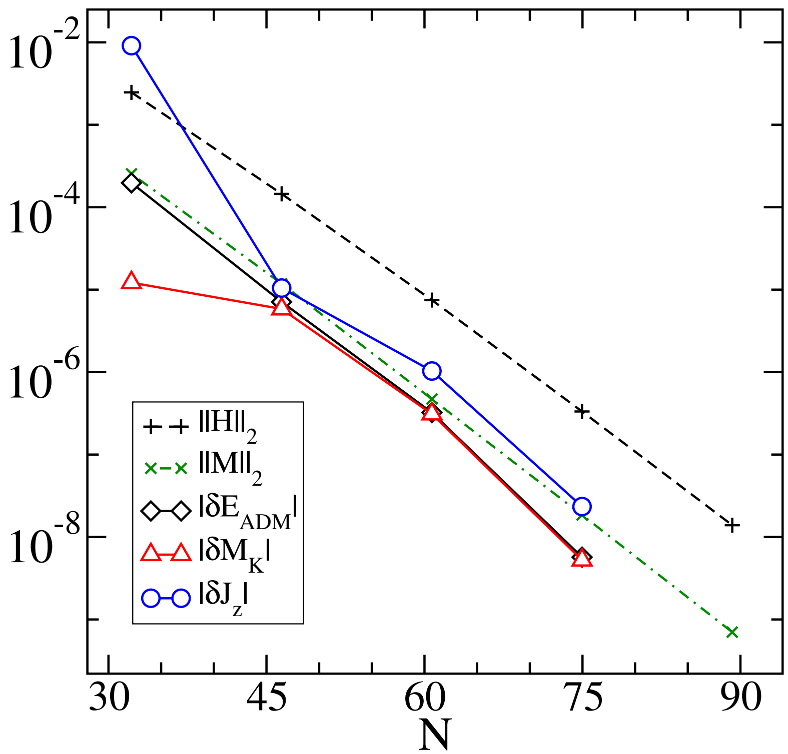

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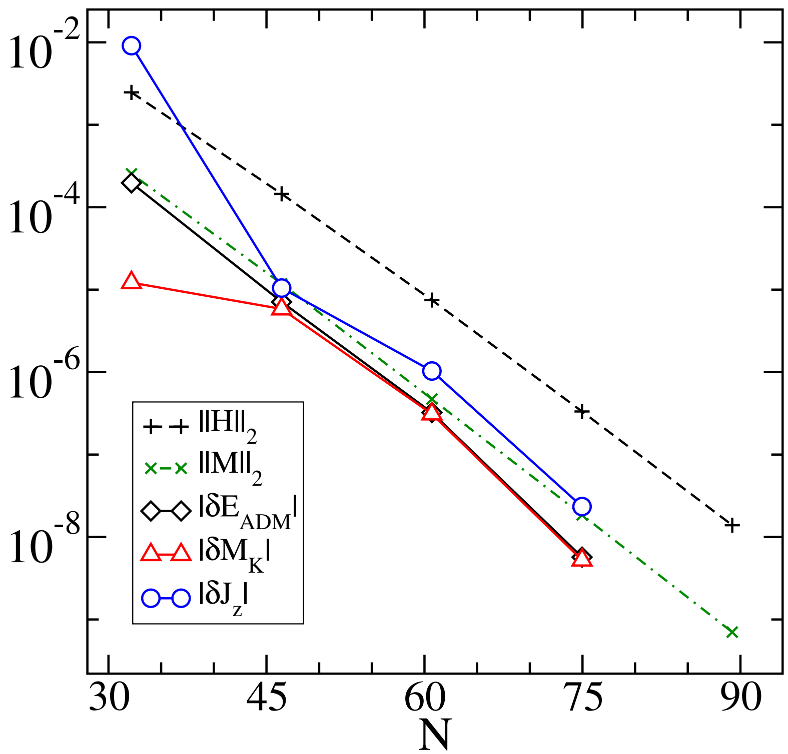


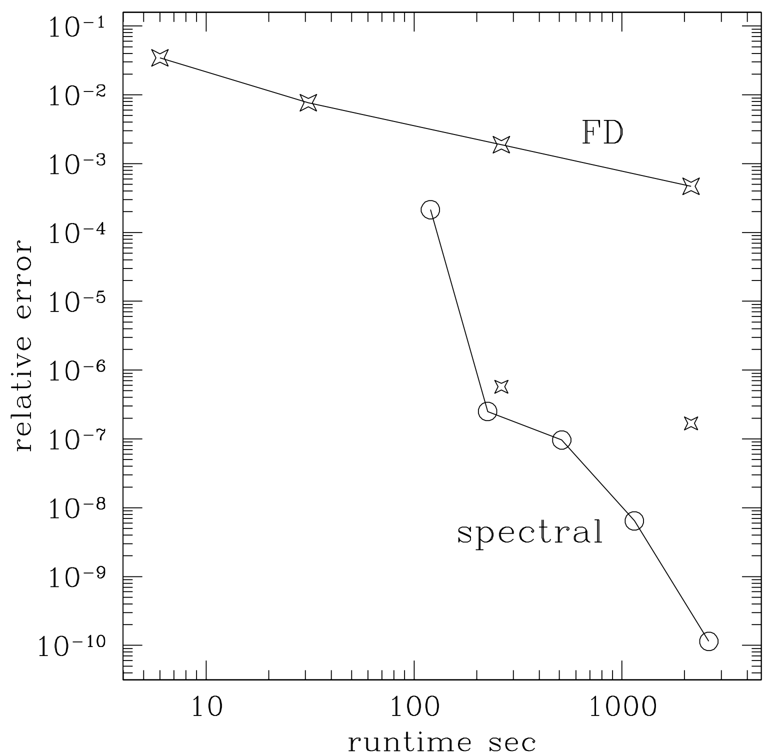


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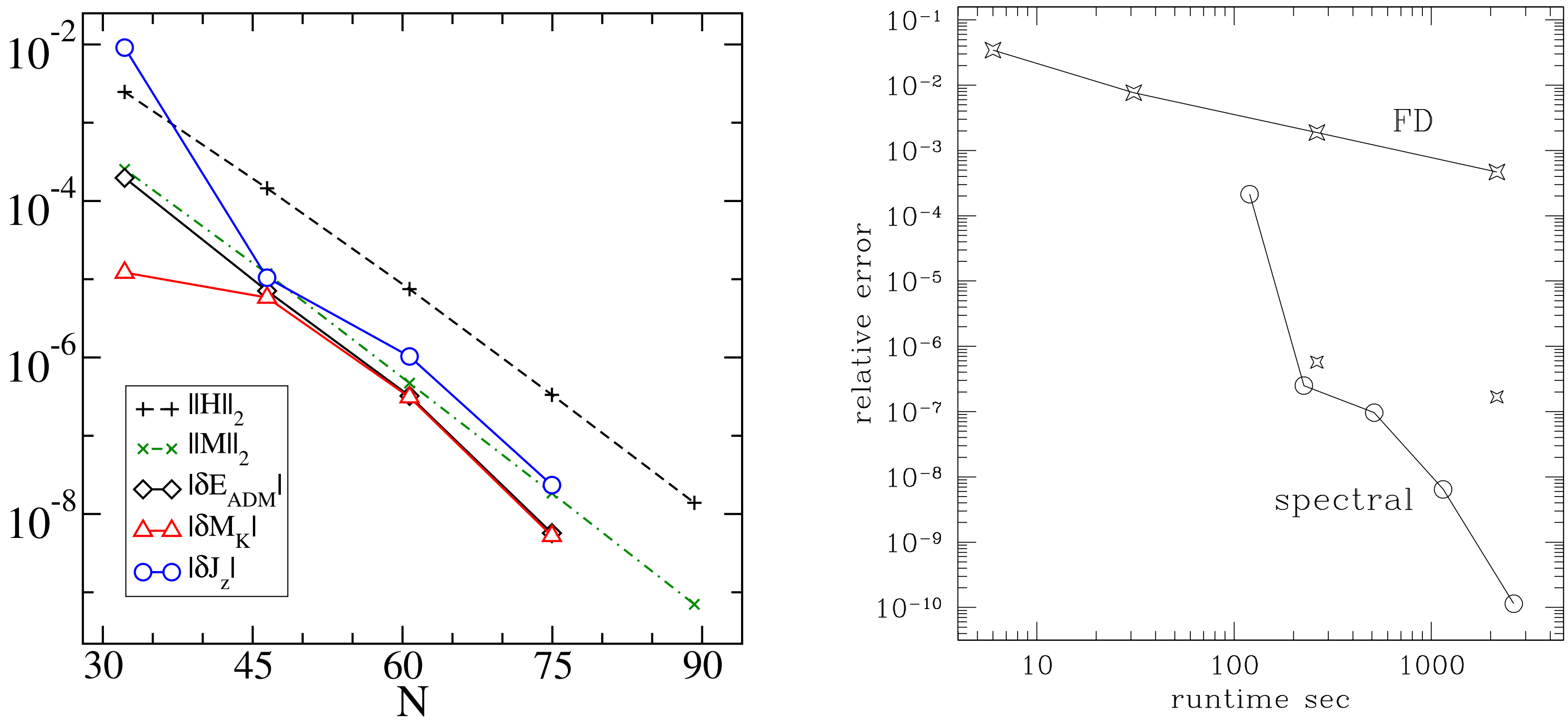
- *Superior efficiency:* Large parameter studies
- *Domain decomposition:* Nontrivial topologies & Multiple length-scales

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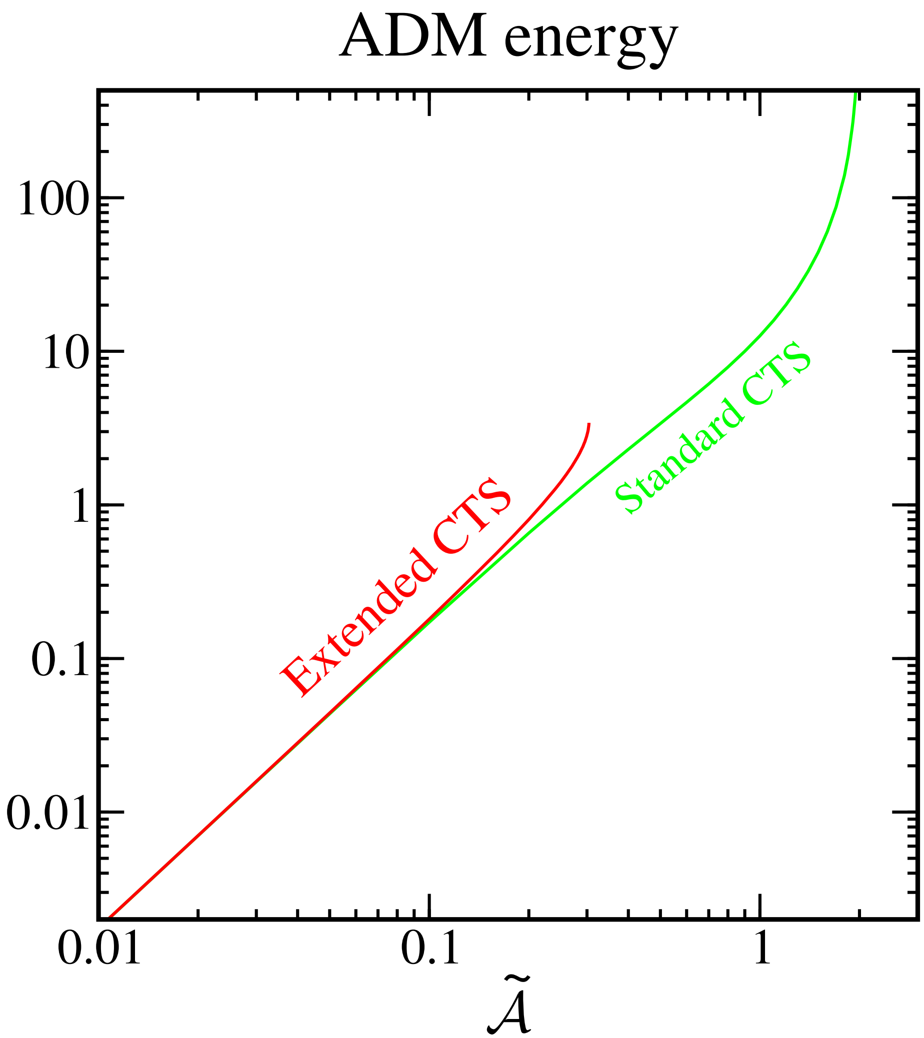
# Non-uniqueness

# Extended conformal thin sandwich equations




HP & York, 2005

$$\tilde{g}_{ij} = \delta_{ij} + \tilde{\mathcal{A}} h_{ij}$$

$$\partial_t \tilde{g}_{ij} = \tilde{\mathcal{A}} \dot{h}_{ij}$$

$$K = \partial_t K = 0$$

(perturbed flat space w/o inner b'dries)

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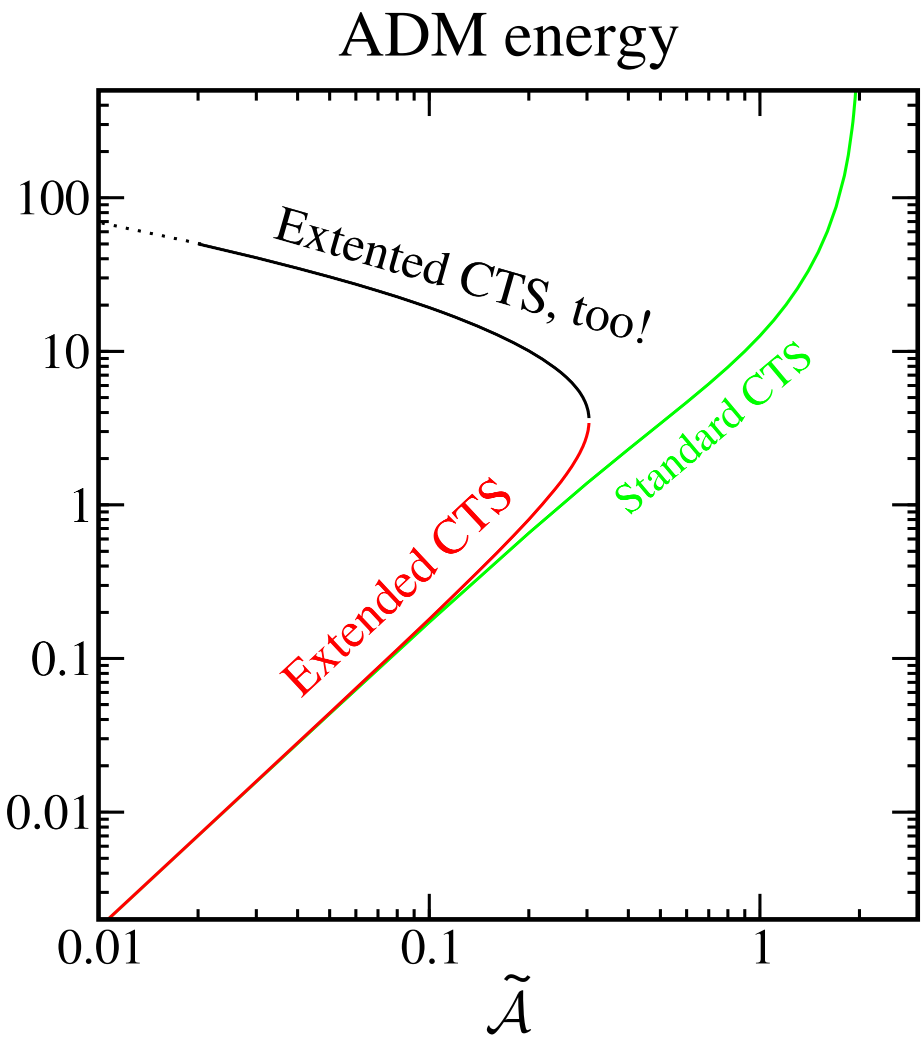


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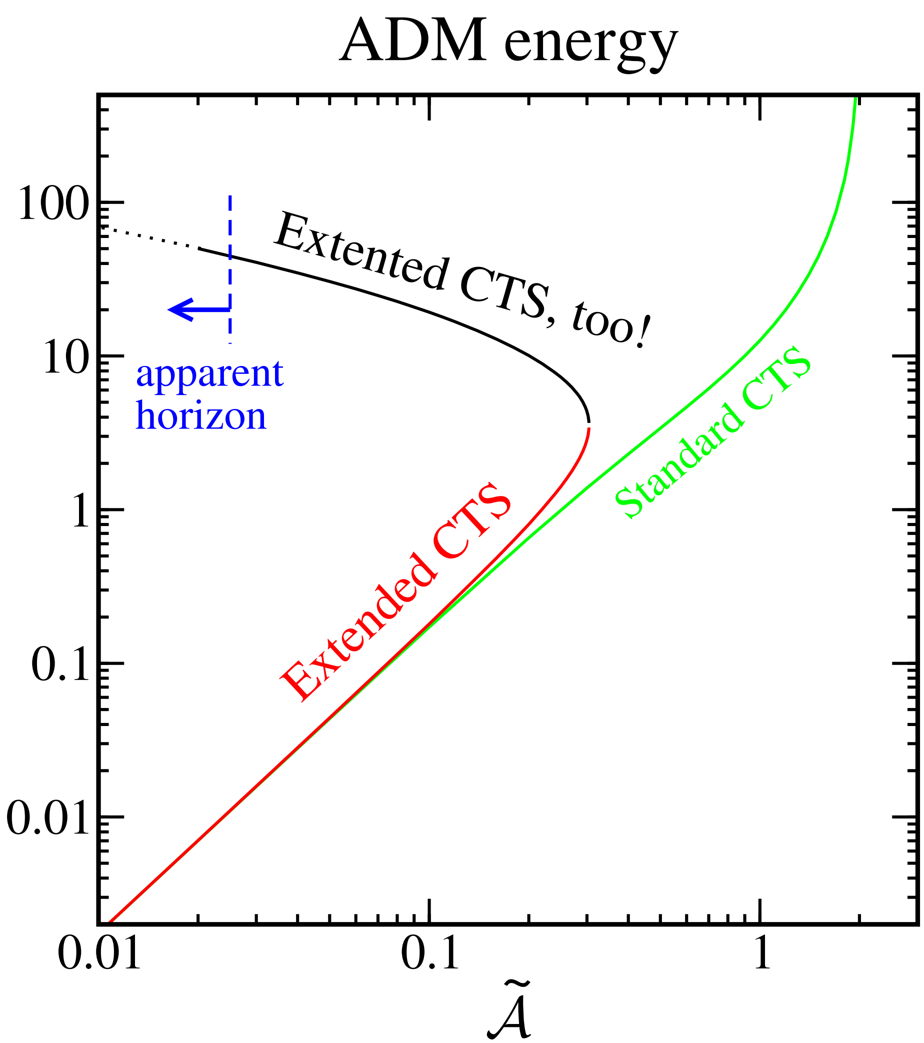


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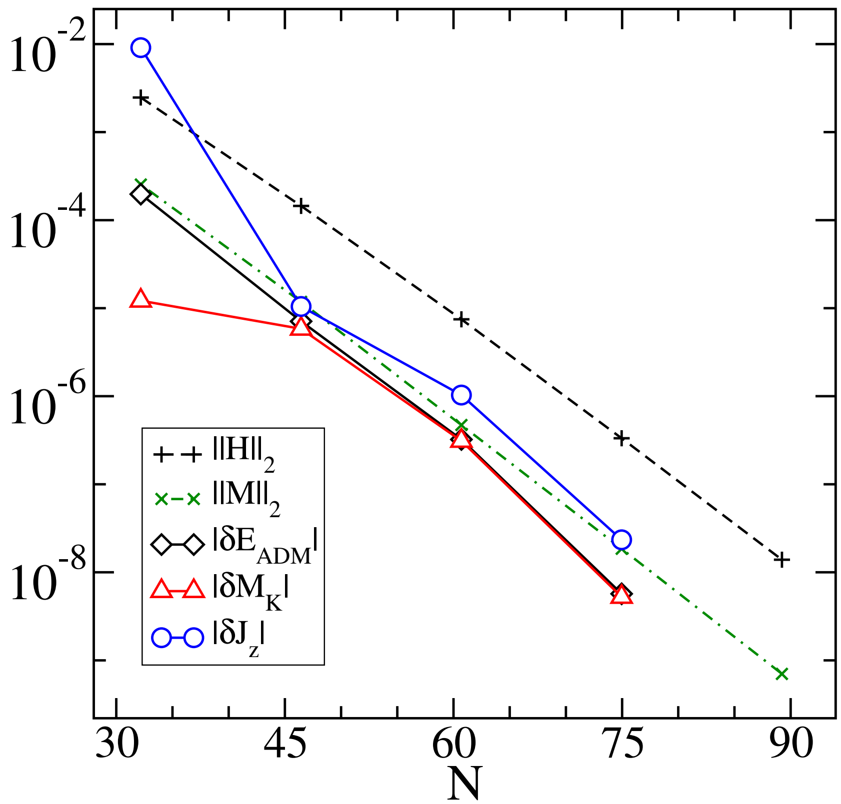
Apparent horizons exist for *small* $\tilde{\mathcal{A}}$!

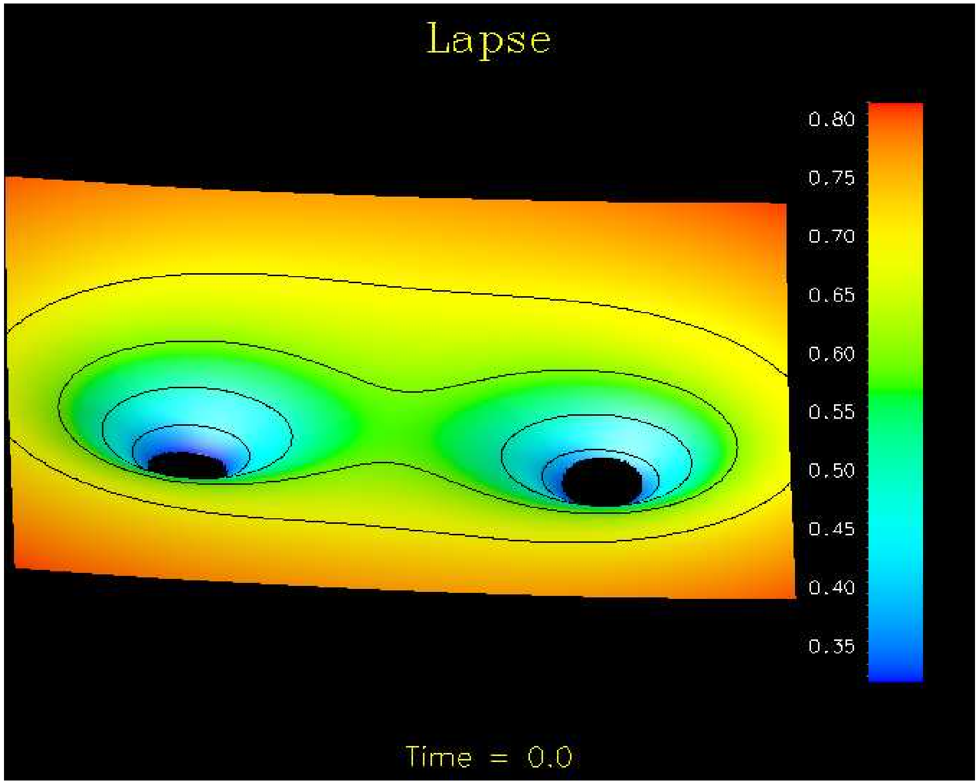
# Properties of QE-ID sets

# Corotating BBH solutions

**Arbitrary choices:** Conformal flatness, $\mathcal{S}$ =sphere. Gauge choices: $K = 0$, $\partial_n(N\psi) = 0$.

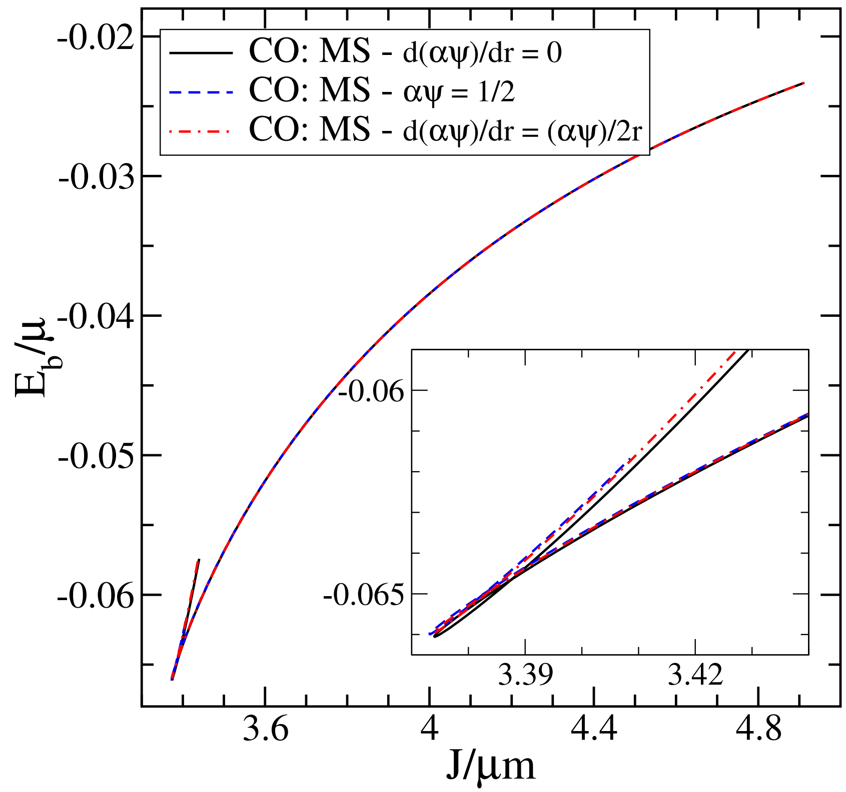


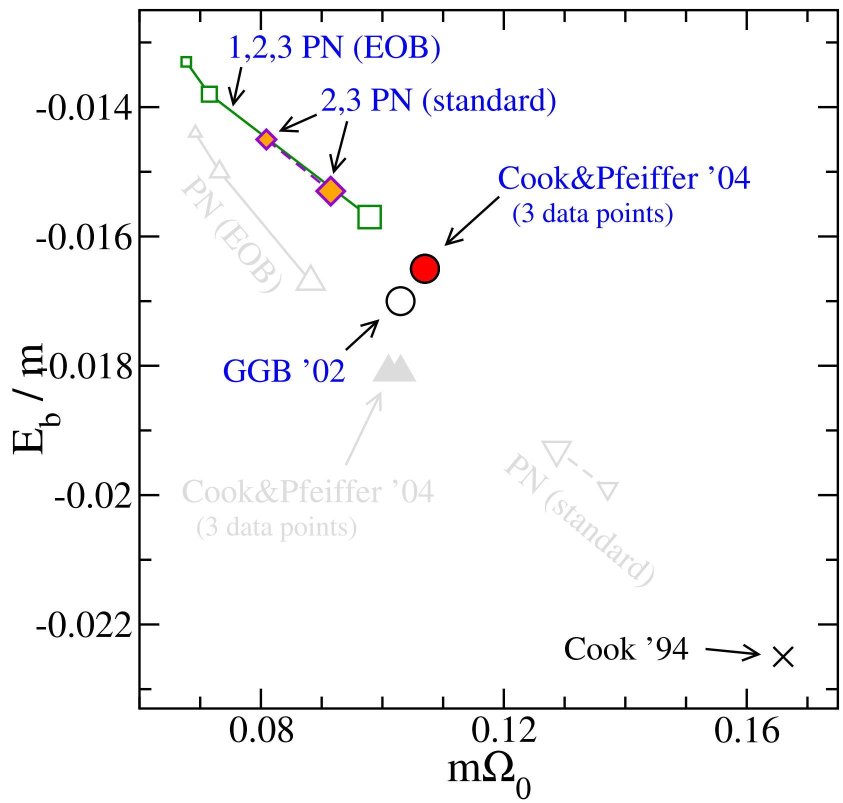

**Exponential convergence**




**Lapse positive through horizon**

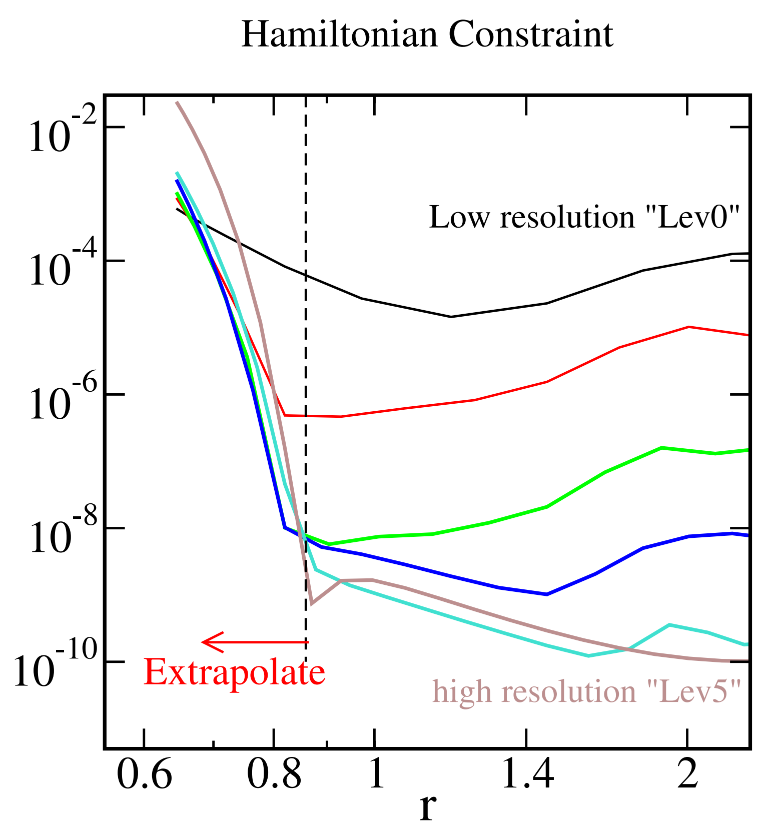
# Sequences of quasi-circular orbits & ISCO



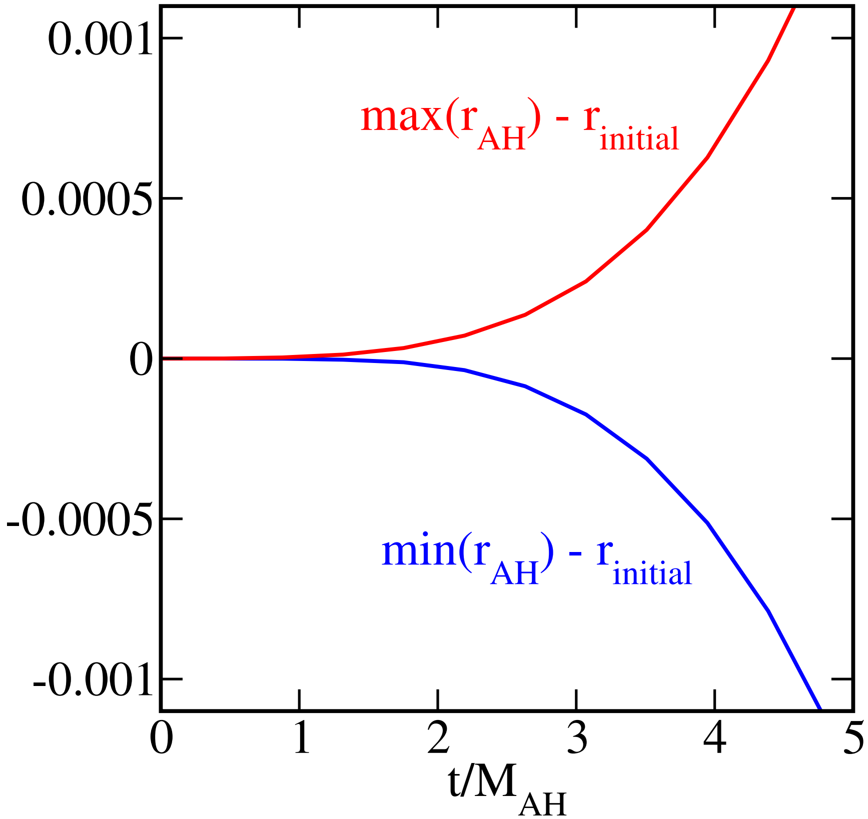
# Towards evolving these ID

- ISCO and other diagnostics very promising
- But, ID only **up to** AH, whereas evolution codes excise **inside** AH
- Extrapolate data inward to $0.75 r_{\rm AH}$
- Constraints violated for $r < r_{\rm AH}$
- The next slides highlight aspects of evolution which are relevant to ID



# Evolution with fixed gauge – horizon motion

Same data as in Mark Scheel's talk – separation 10.




Initially at rest, no transient

**$N$ and $\beta^i$ are excellent initial gauge**

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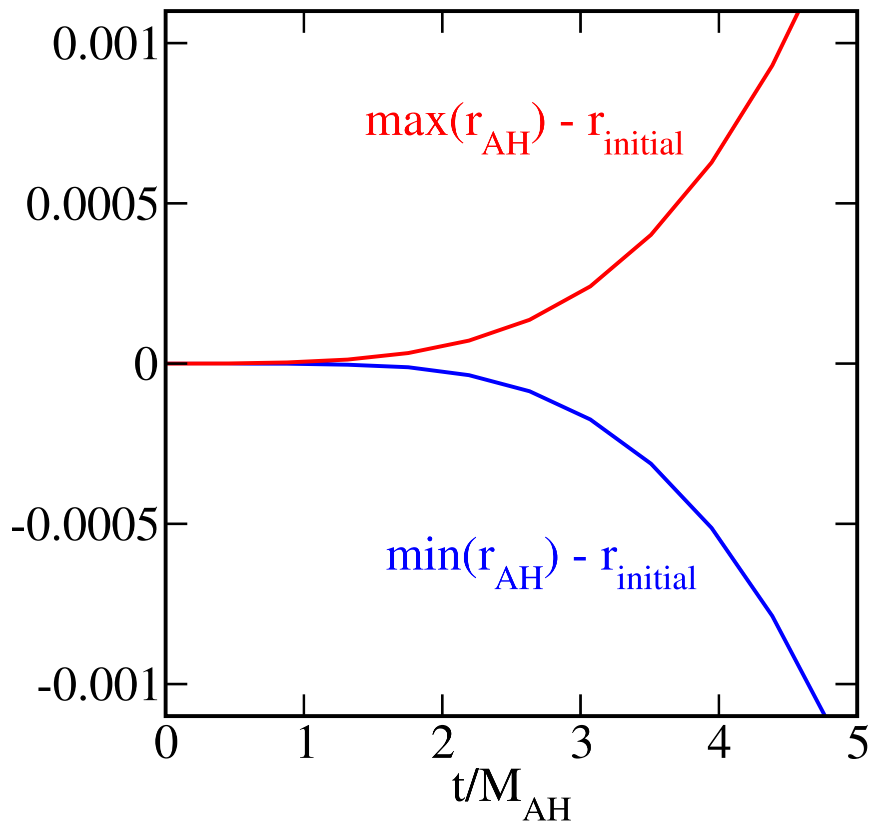


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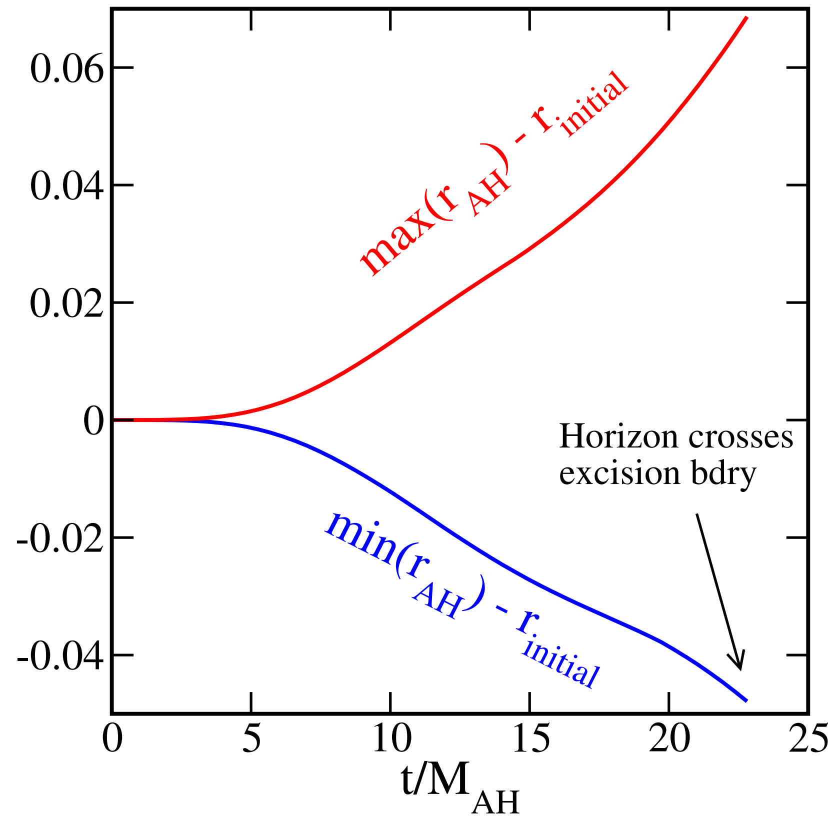
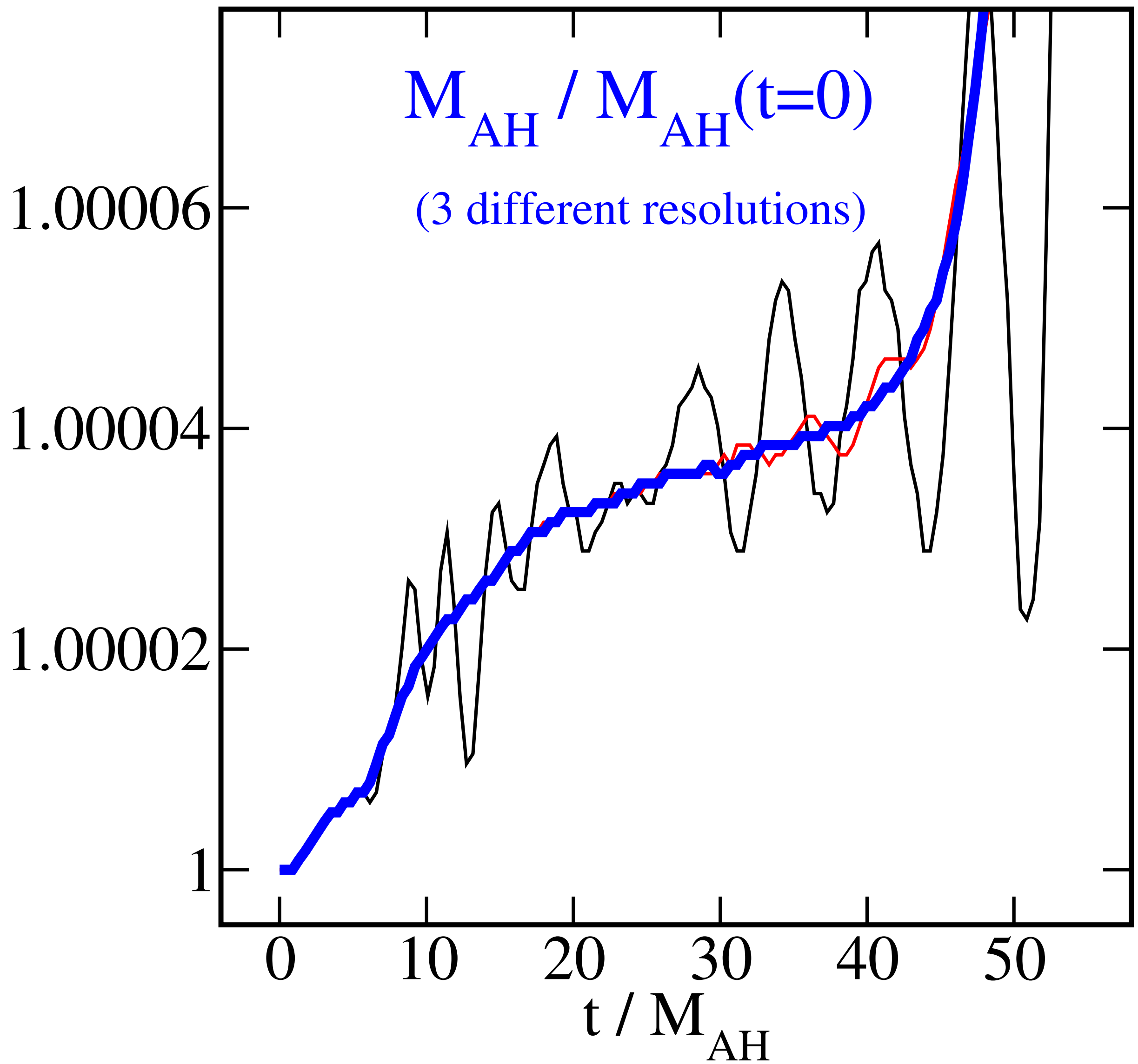


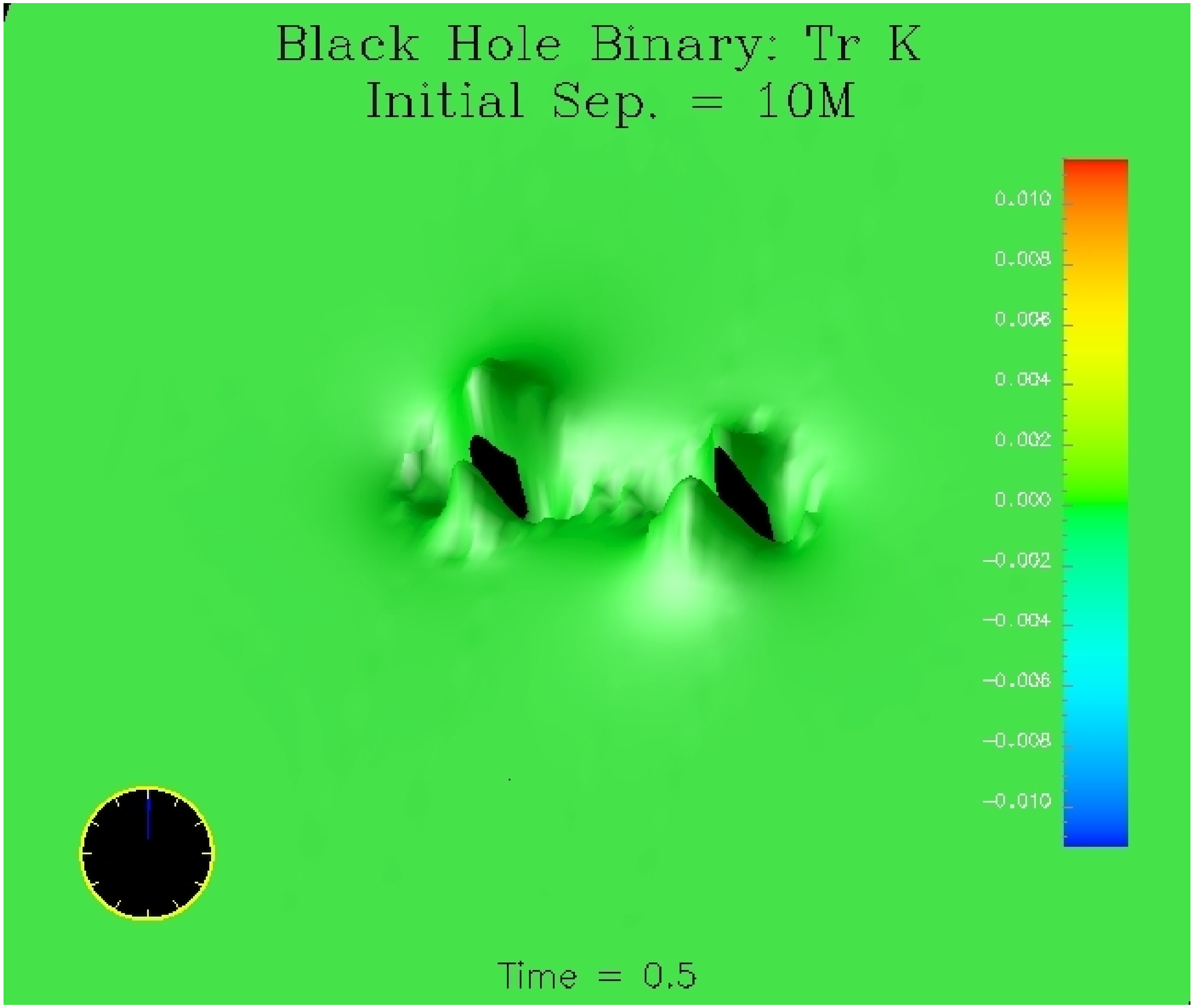

On longer time-scales, AH deforms

**$N$ and $\beta^i$ are excellent initial gauge**

# Apparent horizon mass



# Not all is well – Tidal distortions




**Tidal distortions not captured correctly with current choices for $\tilde{g}_{ij}$ and $\mathcal{S}$**

— Work in progress —

# Public ID repository

# Initial data repository

- http://www.tapir.caltech.edu/~harald/PublicID
- Equal mass BBHs in corotation
- Two choices for Lapse-BC – Eq. (59a) or (59b) from Cook&HP, 2004

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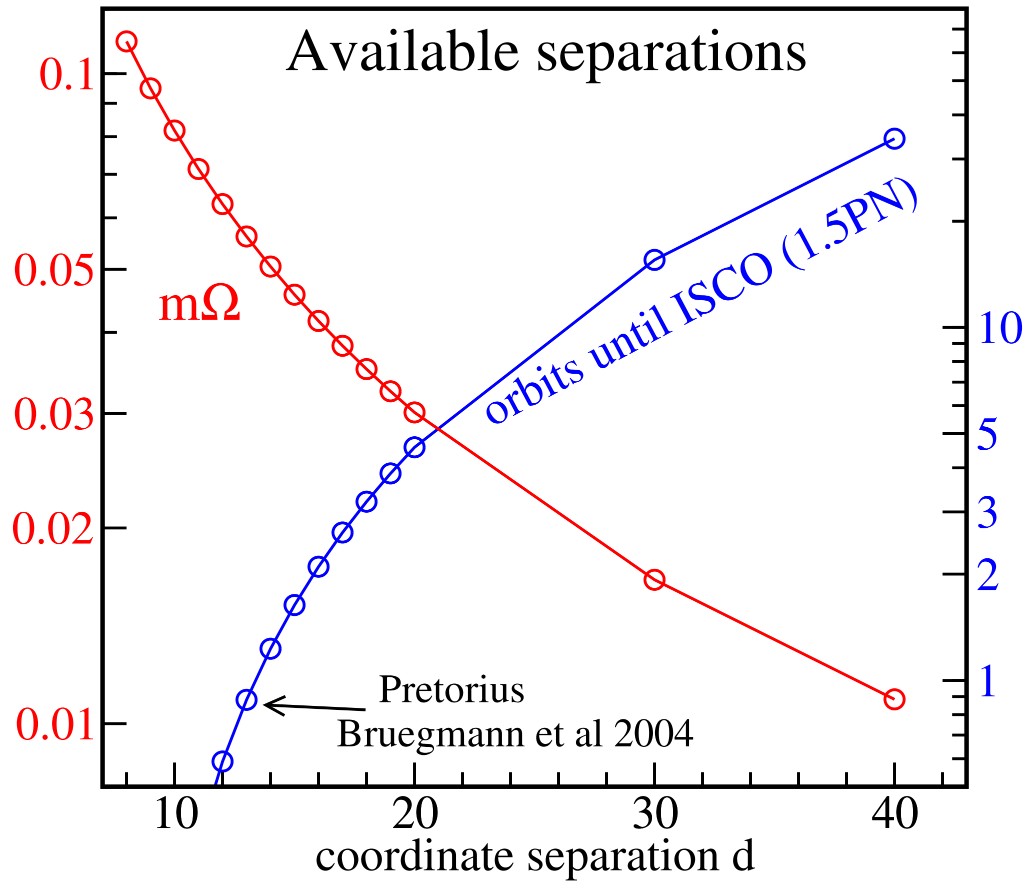

# Using the public QE-BBH initial data

`http://www.tapir.caltech.edu/~harald/PublicID`

The web-site contains:

- Data sets, containing $g_{ij}$, $K_{ij}$, $N$, $\beta^i$ in **Cartesian** components
- Library to interpolate the data to any desired point $(x, y, z)$
  (as long as it is inside the covered computational domain)
- Example executable and example data-set
  (Schwarzschild in Kerr-Schild coordinates)

# Summary

- Framework for BBH initial data in a kinematical setting (helical Killing vector)
- Advantages:
  1. Agreement with PN
  2. $N > 0$, AH initially constant, $M_{\rm AH}$ exceedingly constant
- Tidal distortions not yet captured
- **Data sets publicly available**
  `http://www.tapir.caltech.edu/~harald/PublicID`
  1. Compute waveforms!
  2. Compare and validate evolution codes on the <u>same</u> initial data

# Contents of a data set

1. **The data** in several resolutions (Lev2, ... Lev5), each in its own subdirectory

2. The file **Convergence** listing errors for each resolution:

```
#....N  Nor-Linf    Nor-L2   Ham-Linf    Ham-L2   Mom-Linf   Mom-L2
32.184    0.2280   0.03185     0.0339   0.00202   0.00552  0.000217  <-- Lev0
46.447 0.0001337 2.452e-05    0.00333  0.000119  0.000461  1.03e-05  <-- Lev1
60.706 4.361e-06 9.697e-07   0.000238  6.12e-06  2.80e-05  4.14e-07  <-- Lev2
74.963 1.432e-07 3.253e-08   1.40e-05  2.71e-07  1.48e-06  1.62e-08  <-- Lev3
89.219 4.855e-09 6.189e-10   7.38e-07  1.13e-08  6.98e-08  6.11e-10  <-- Lev4
103.47 3.065e-10 1.267e-11   3.59e-08  4.45e-10  3.08e-09  2.24e-11  <-- Lev5
        ^^^^^^^^^^^^^^^^^    ^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^

    Change between this and    Hamiltonian and momentum constraints
    next lower resolution      outside horizon
```

3. The file **AdmQuantitites** listing some relevant quantities at each resolution

```
#....N             J_ADM[2]            Sph-Exp-MAH            EADM_corr
32.184       4.4323659273442       1.14349189174678       2.250487955424110     <-- Lev0
46.447       4.4405908812160       1.14359857991925       2.250608439097928     <-- Lev1
60.706       4.4405869341762       1.14360434438628       2.250628605804796     <-- Lev2
74.963       4.4405875539439       1.14360453955136       2.250630317404318     <-- Lev3
89.219       4.4405875462573       1.14360454280063       2.250630342690472     <-- Lev4
103.47       4.4405875503605       1.14360454285761       2.250630340509140     <-- Lev5
```

4. The file **Omega** containing the orbital angular frequency

# Interpolation Library – suggestions welcome!

- **Library** libSpECLibraryID.a (compiled with gcc 3.4.3 on RHE 9)
- **Header file** PublicID.hpp:

```
#include <vector>

void ReadData(const double Omega);  // import from disk

void InterpolateData(const vector<double>& x,
                     const vector<double>& y,
                     const vector<double>& z,
                     vector<double>& gxx,   ... , vector<double>& gzz,
                     vector<double>& Kxx,   ... , vector<double>& Kzz,
                     vector<double>& Betax, ... , vector<double>& Betaz,
                     vector<double>& N);

void ReleaseData(); // free memory
```

- **Test-executable** InterpolateExample.cpp:

```
g++ InterpolateExample.cpp libSpECLibraryID.a -lblas
```