Solving Einstein's equations using spectral methods

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Outline:

1. Numerical methods: pseudospectral multidomain evolution.
2. Binary black hole evolutions using the KST formulation.
3. A new generalized harmonic formulation.

I. Numerical Methods:
Pseudospectral Multidomain Evolution

Use pseudospectral discretization

- Write **solution** as sum of N basis functions $\phi_k(x)$: $u(x,t) = \sum_{k=0}^{\infty} \tilde{u}_k(t)\phi_k(x).$

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  - For smooth $u(x,t)$, error $\sim e^{-N}$.
- Construct discrete inverse transform: $\quad \tilde{u}_k(t) = \sum_{n=0}^{N-1} w_n u^{(N)}(x_n,t)\phi_k(x_n).$
  - Requires careful choice of collocation points $\{x_n\}$.
  - Can transform at will (*with no error!*) between $u^{(N)}(x_n,t)$ and $\tilde{u}_k(t)$.

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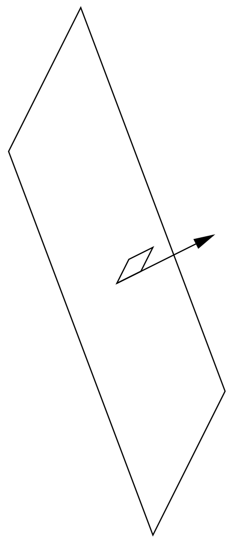
- To solve nonlinear hyperbolic PDEs:
  - Compute derivatives in spectral space.
  - Compute nonlinear terms in physical space.
  - Time integration via method of lines.
  - Boundary conditions imposed analytically on characteristic fields (so excision is trivial).

Characteristic decomposition

- Einstein evolution equations written in first-order hyperbolic form.

- Hyperbolicity guarantees complete set of
  - **characteristic fields** $u^{\hat{\alpha}}$
  - **characteristic speeds** $v_{(\hat{\alpha})}$

Characteristic decomposition



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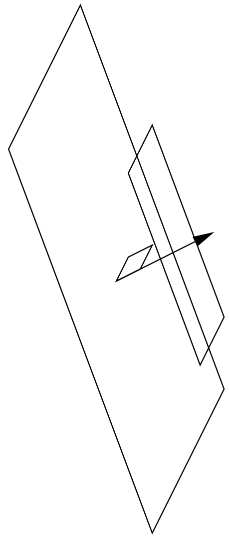

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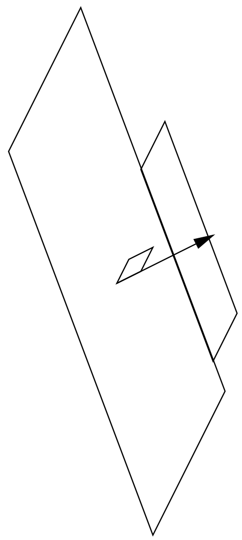

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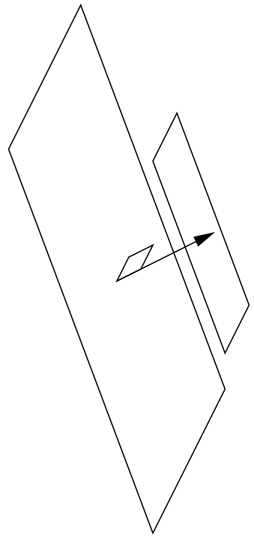

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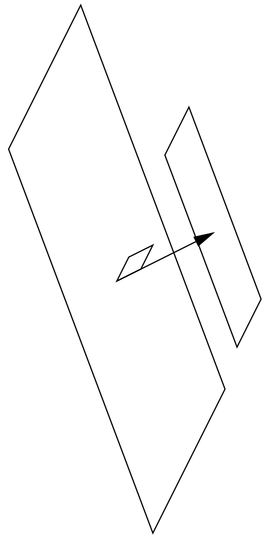

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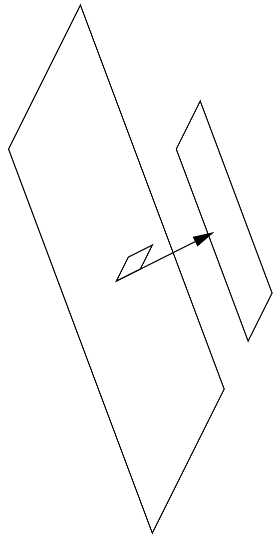

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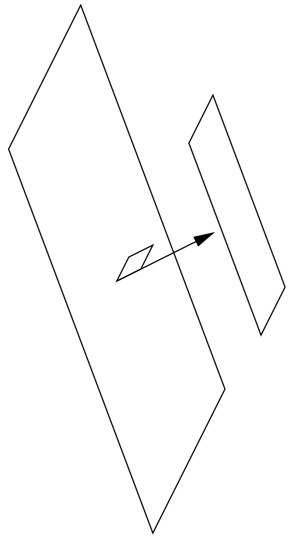

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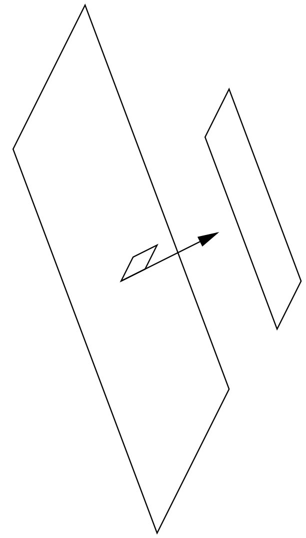

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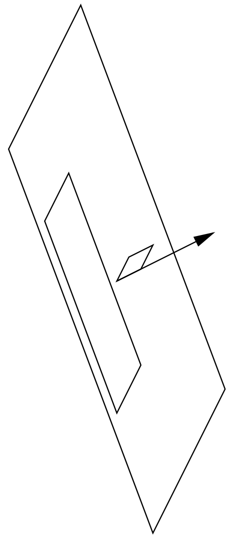

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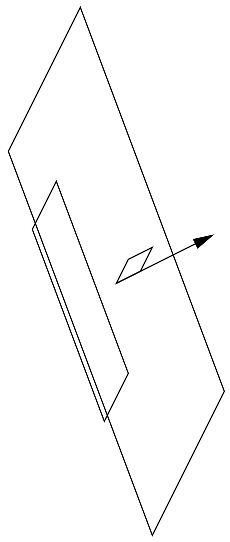

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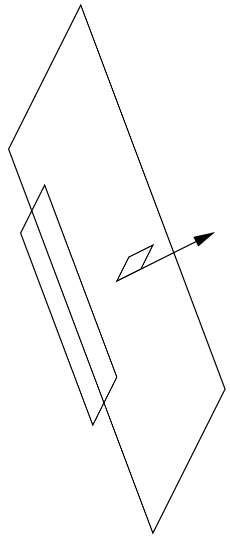

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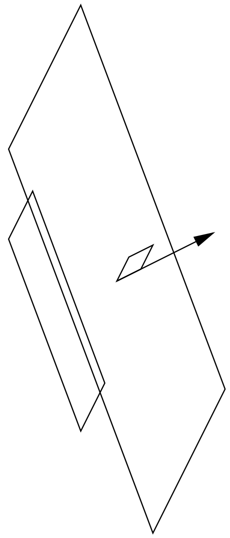

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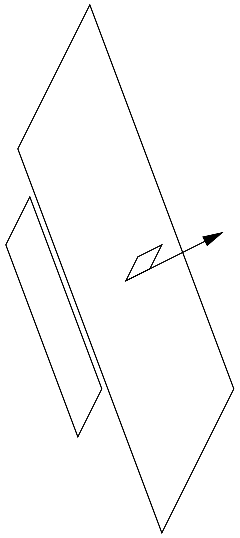

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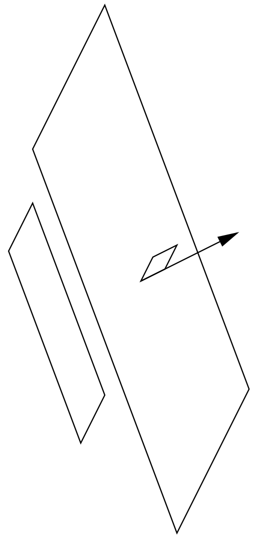

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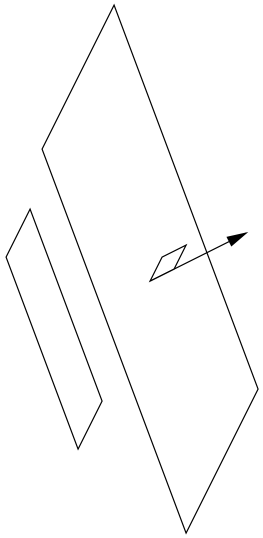

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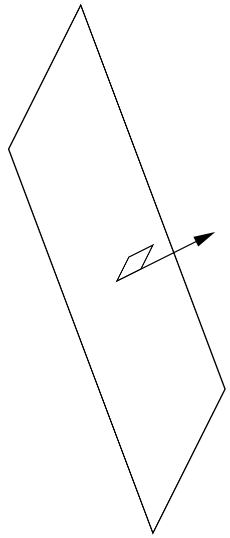

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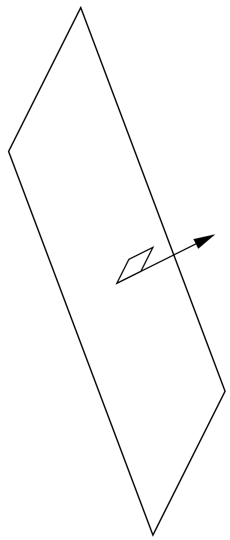

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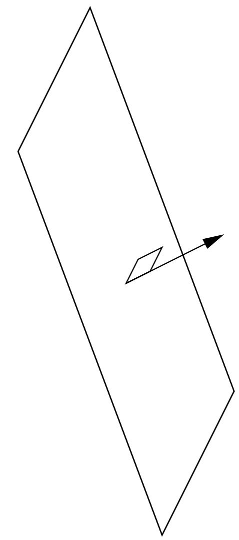

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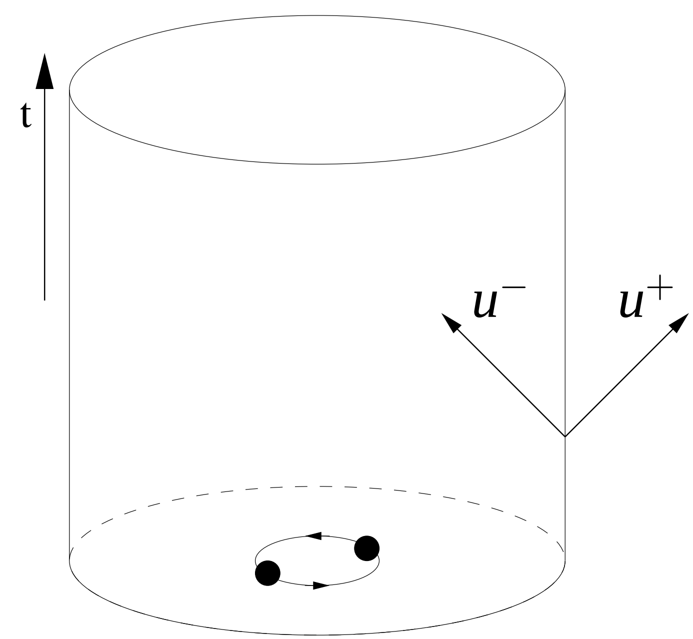

- Boundary conditions required on **all incoming** ($v_{(\hat{\alpha})} < 0$) characteristic fields.

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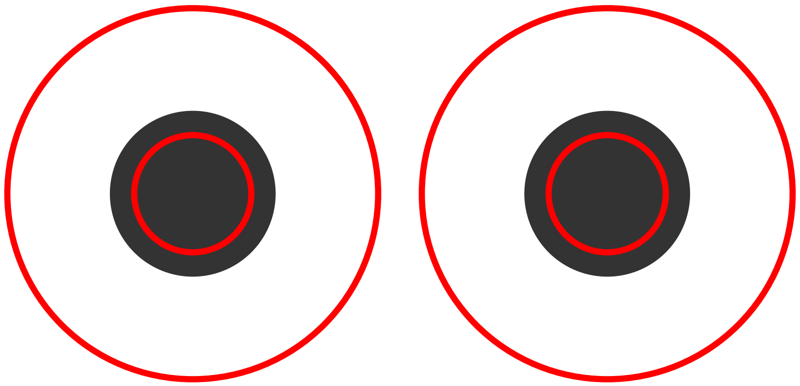

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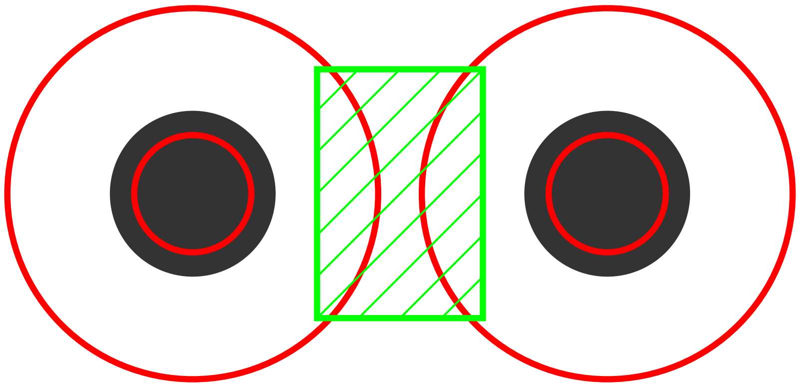

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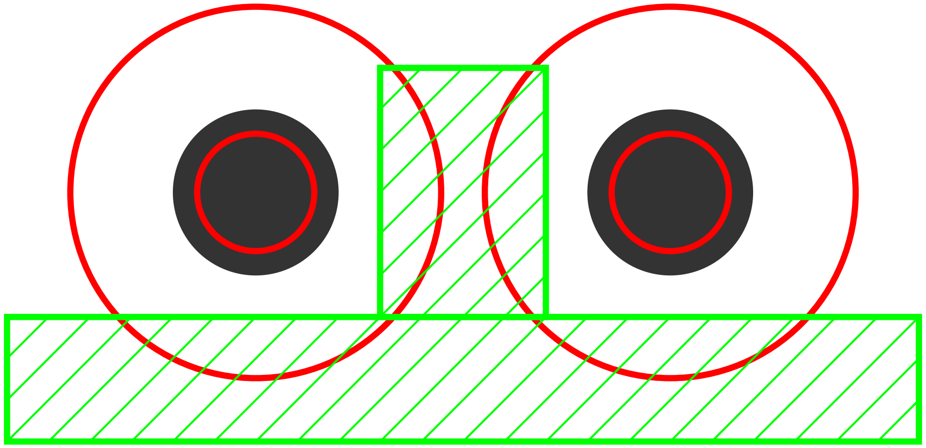

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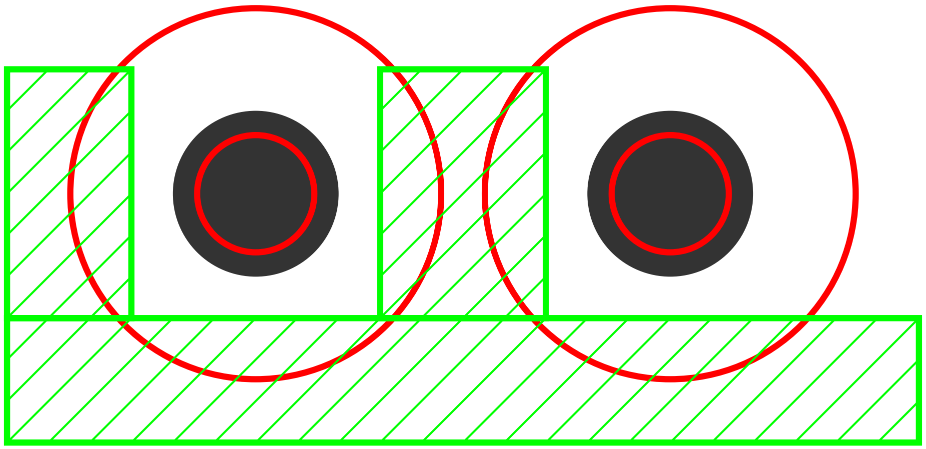

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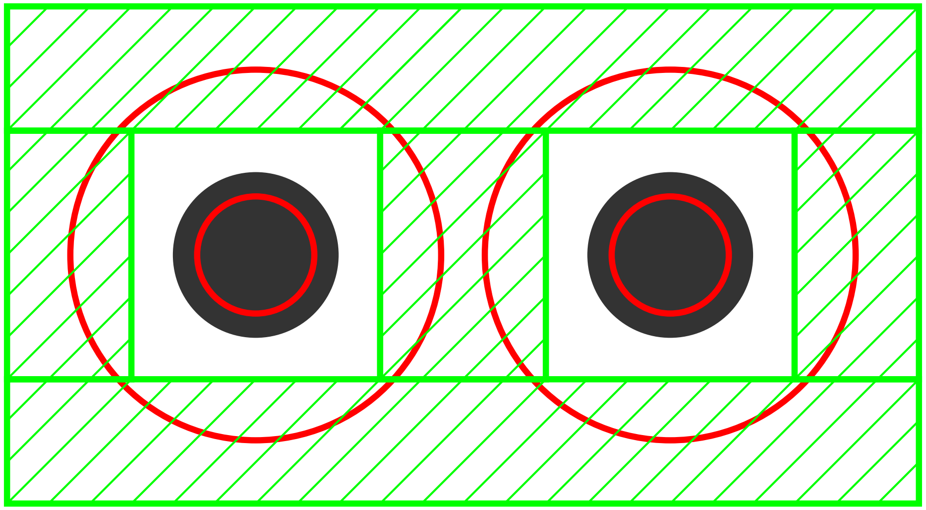

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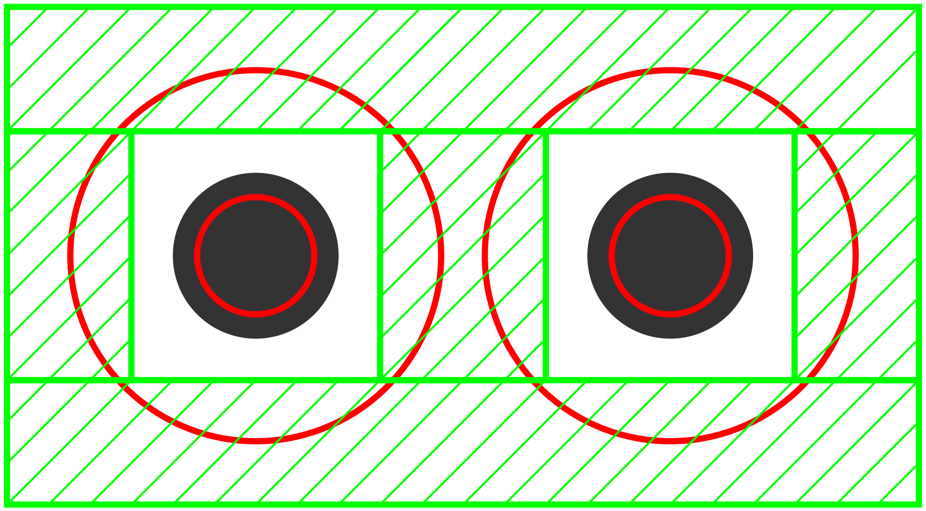
- Subdomains evolve 'independently'

$\Rightarrow$ Natural parallelism



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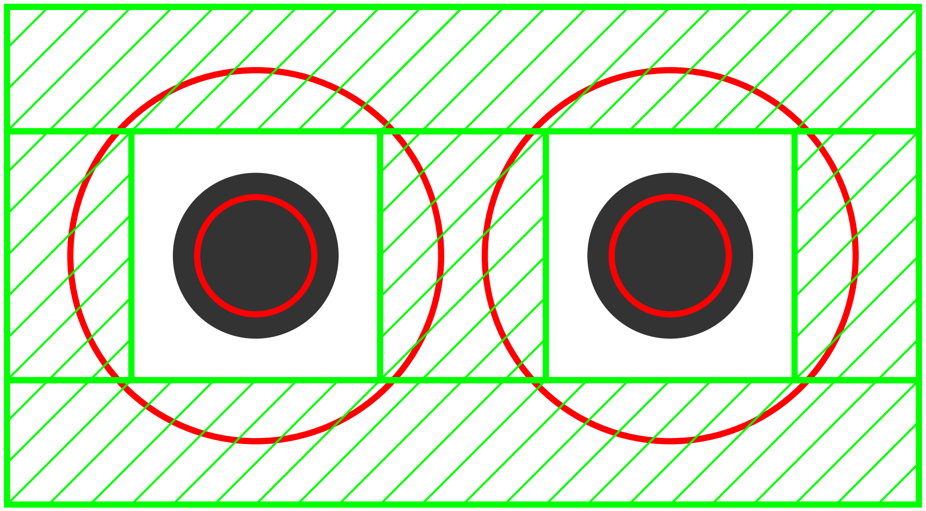

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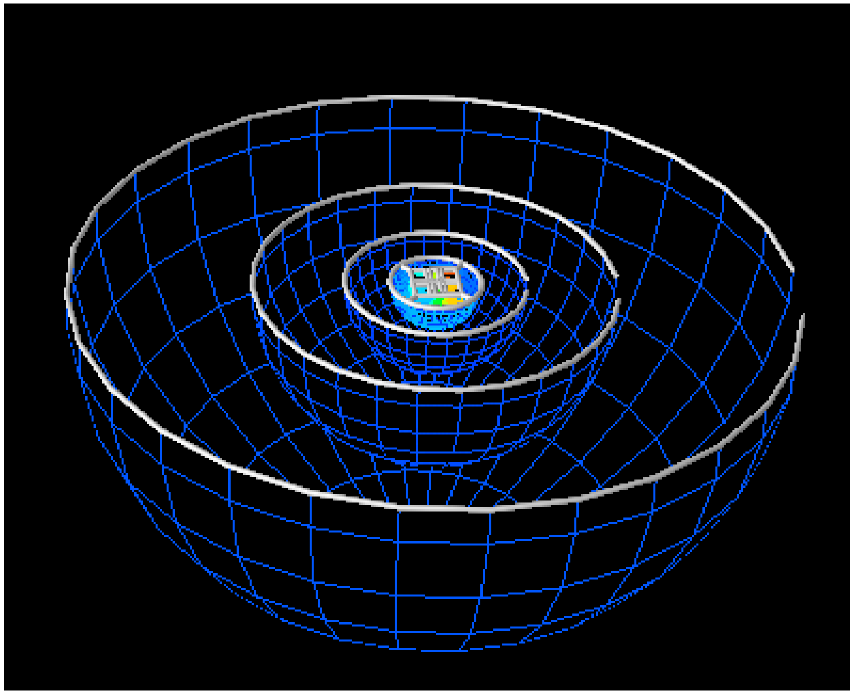
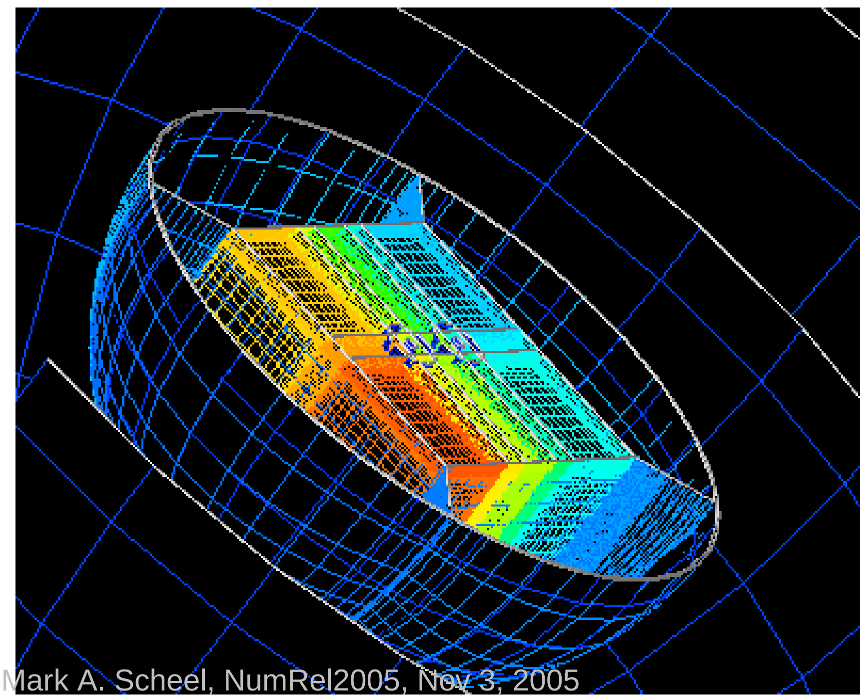
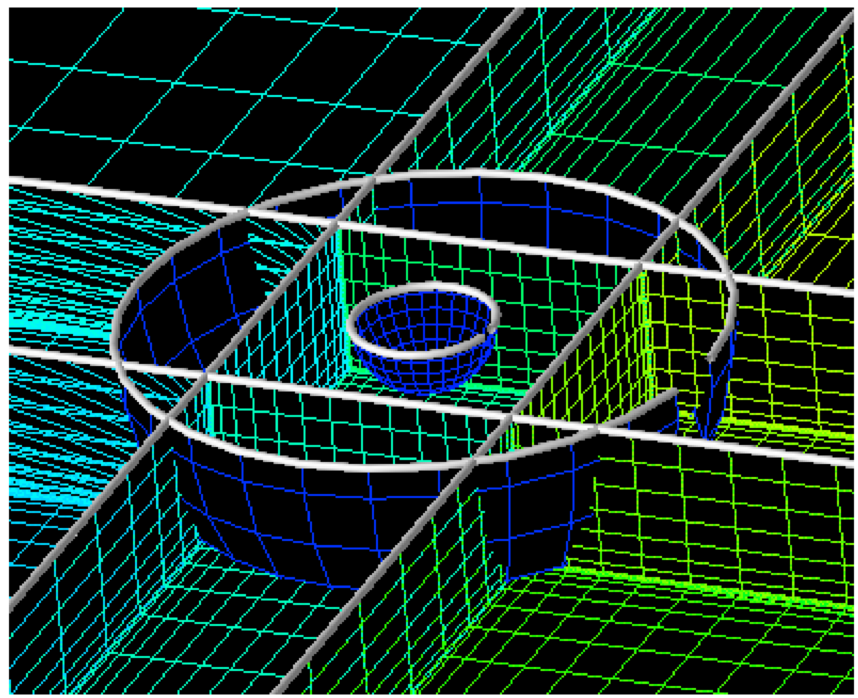
- Boundary conditions: fill ingoing characteristic fields from neighbor.

Domain decomposition



- 54 subdomains:
  - 2 inner spherical shells (1 per BH)
  - 43 rectangular subdomains.
  - 6 subdomains $= 1$ 'cubed sphere'
  - 3 outer spheres (to $r_{\max} = 320 M_{BH}$)





II. Binary Black Holes using KST

KST evolution: Basic setup

- Evolution: Free evolution, KST formulation.
  Kidder,Scheel,Teukolsky PRD 64, 064017 (2001)

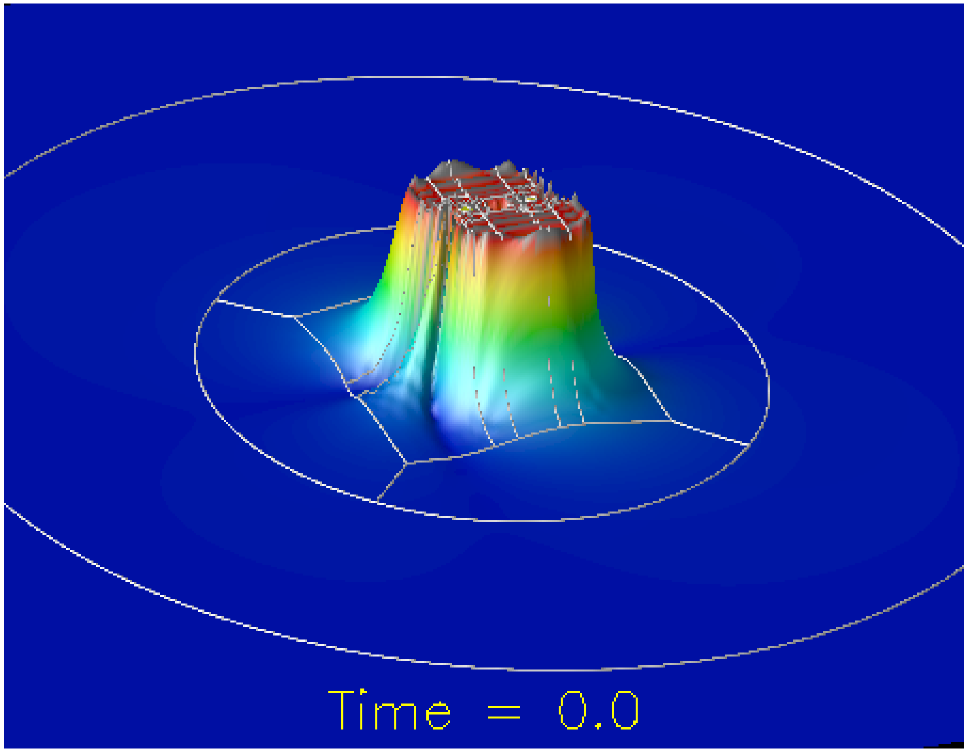
- Initial Data: QE Conformal Thin Sandwich (See Harald Pfeiffer's talk)
  - `sep_10.00_59a.tgz` from `http://www.tapir.caltech.edu/~harald/PublicID`
  - Orbital period $156 M_{BH}$

- Boundary Conditions:
  - No boundary condition at horizons (none needed!)
  - Constant-incoming-characteristic-BCs (like Sommerfeld) at $r = 320 M_{BH}$.

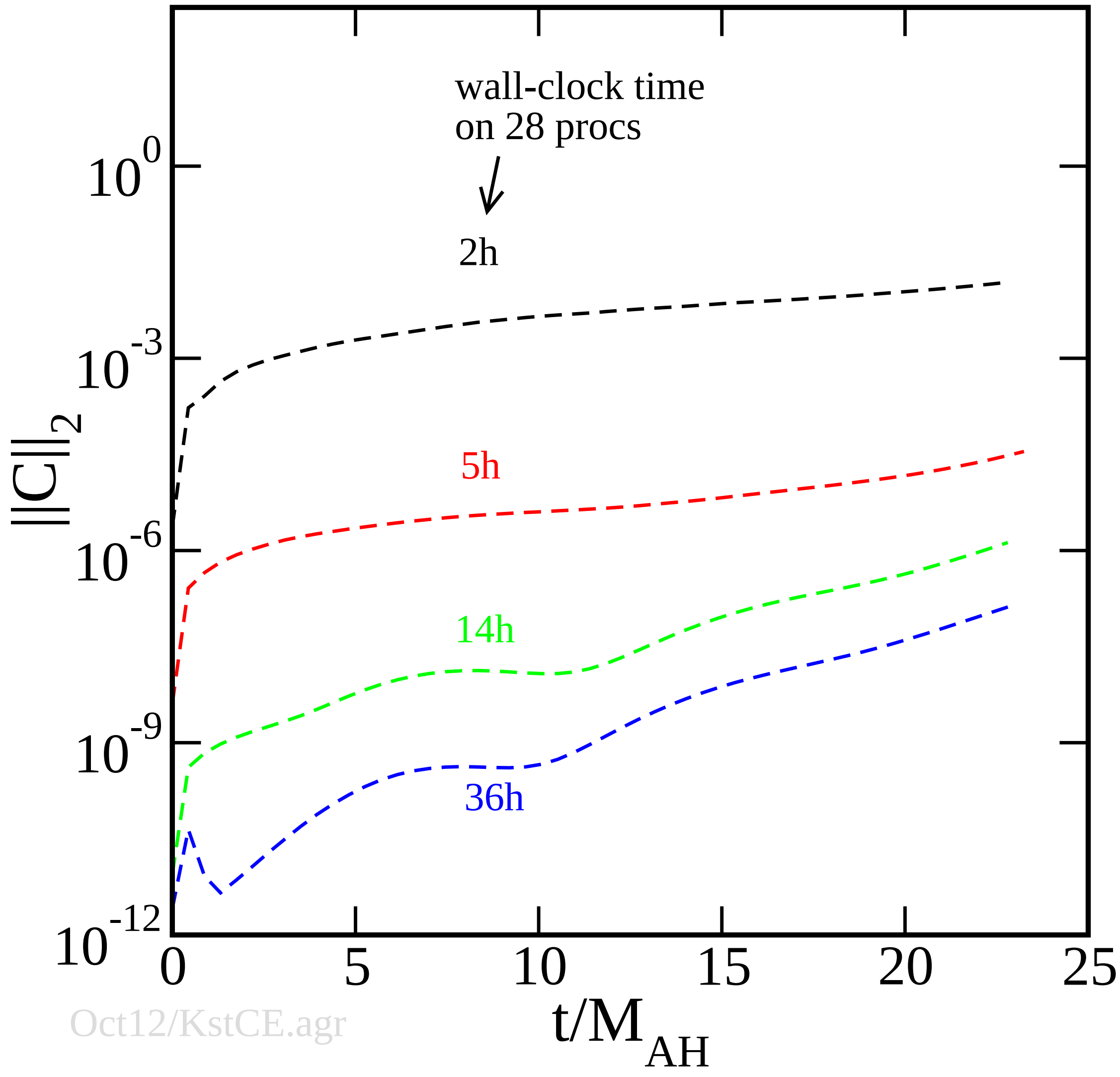
- Gauge Conditions:
  - Initial data gives lapse, shift in corotating frame.
  - Shift and densitized lapse held constant in time.

KST evolution: Psi4

- Movie of $|\Psi_4|$.



KST evolution: Constraint errors



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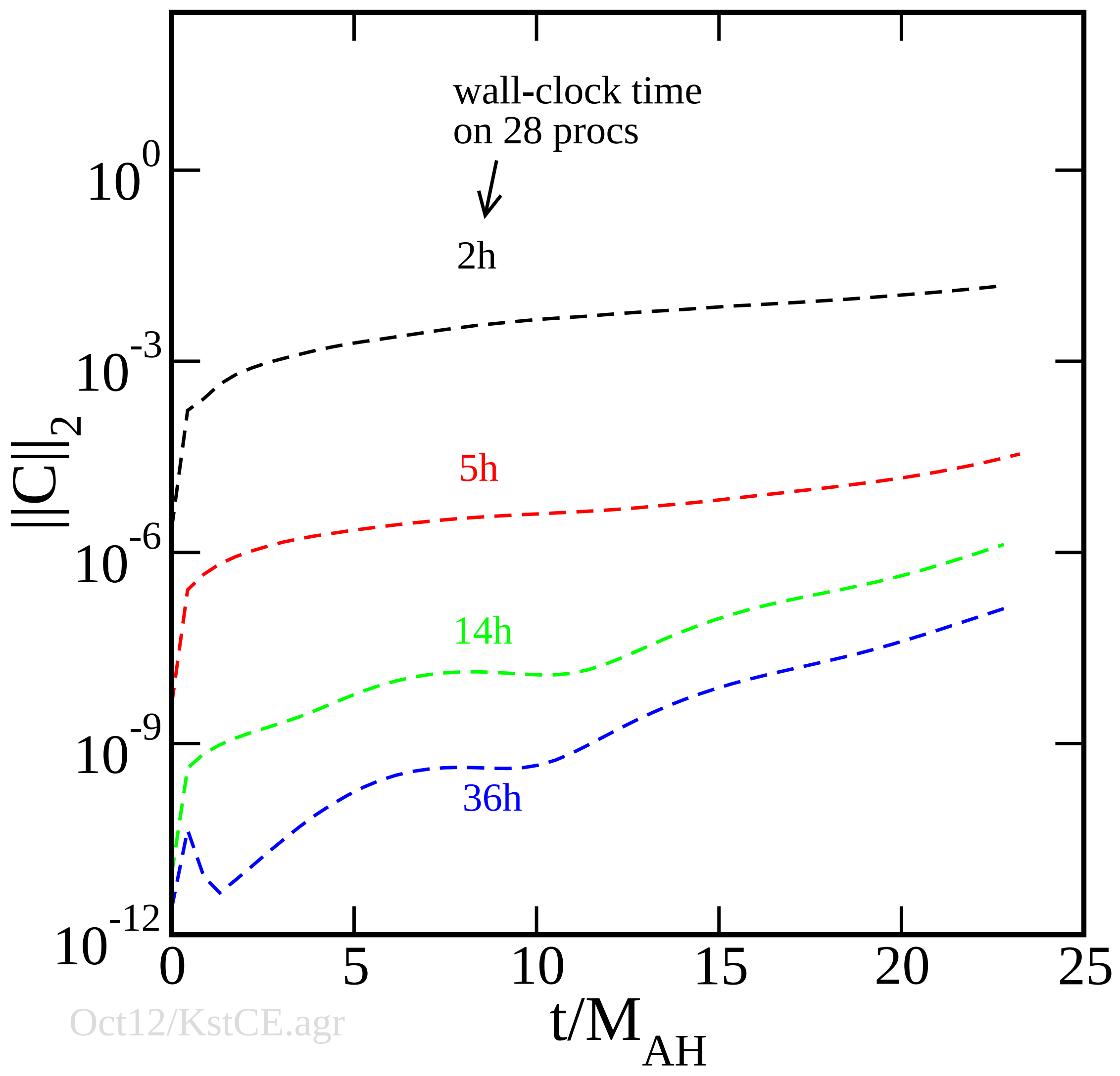
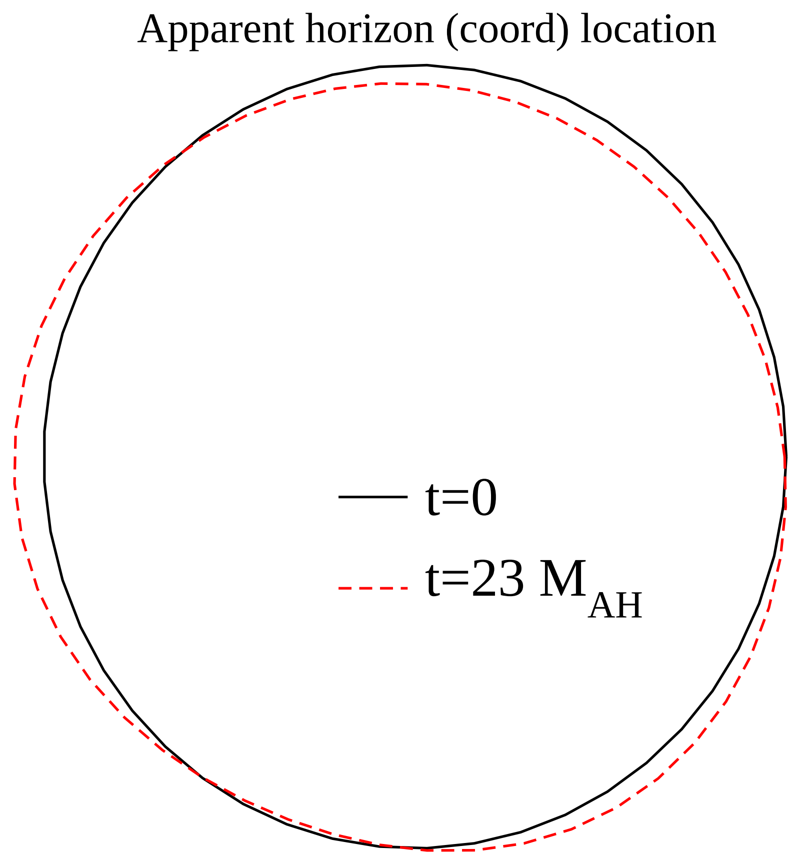
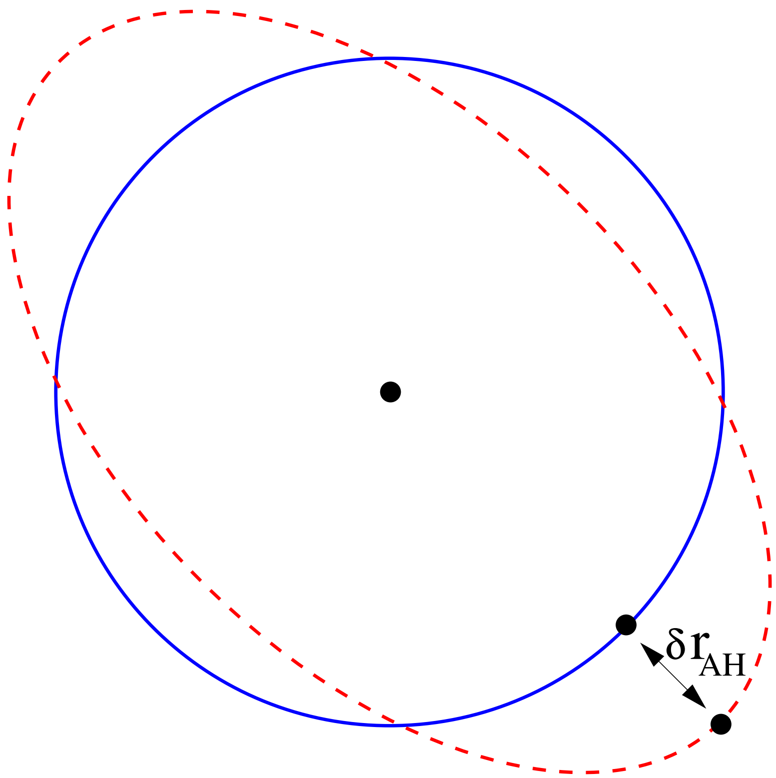




KST evolution: Shift adjustment

- Purpose: Keep all AHs at constant coordinate locations.
- To implement near each hole:
  - Set $\beta^i = \beta^i_{\mathrm{initial}} + \delta r_{AH}(\theta, \varphi) f(r)(x^i/r)$
  - Choose $f(r)$ so adjustment falls off away from hole.
- Apply adjustment every $\delta t_{\mathrm{adjust}} = 0.5 M_{BH}$.



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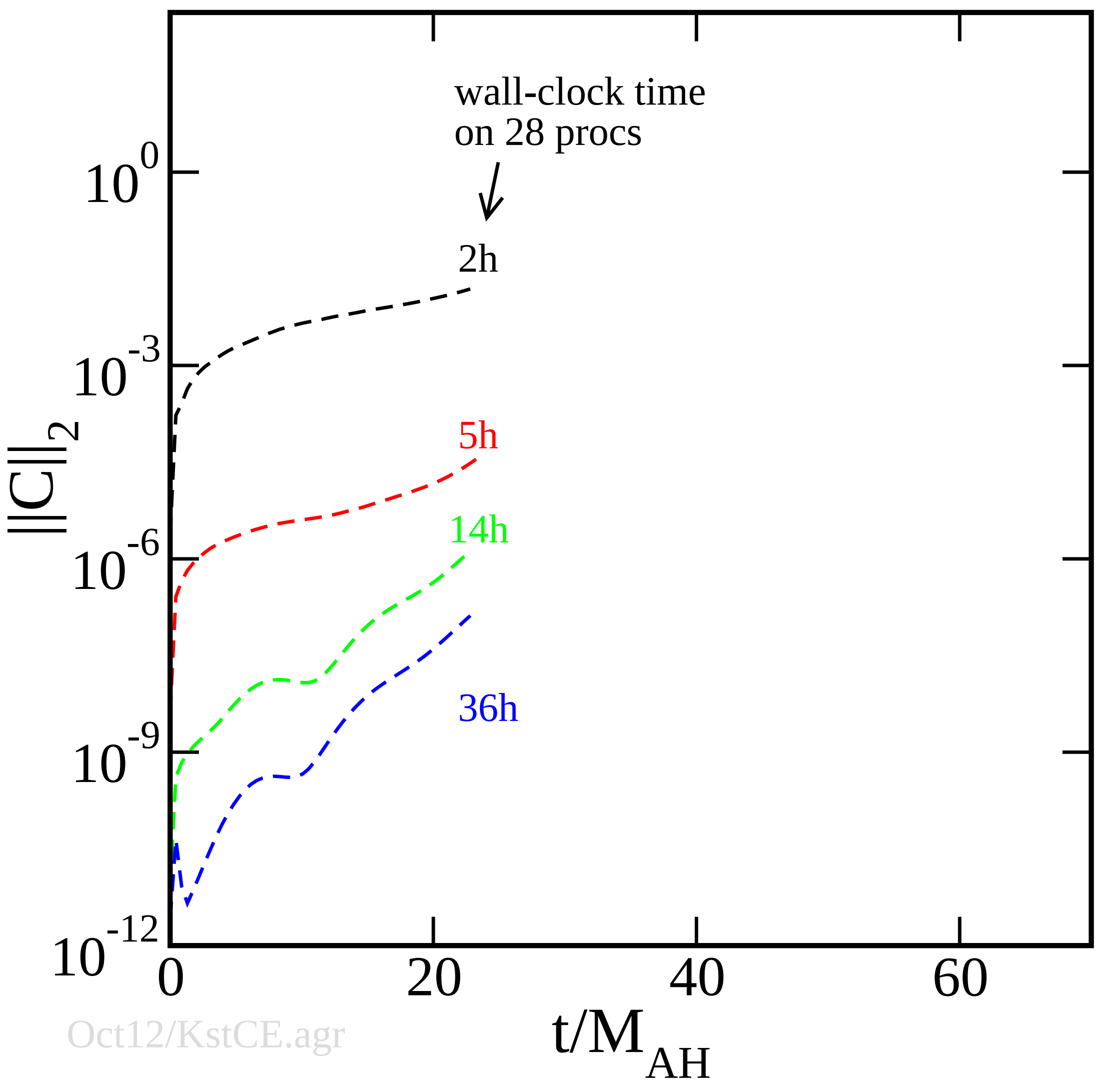

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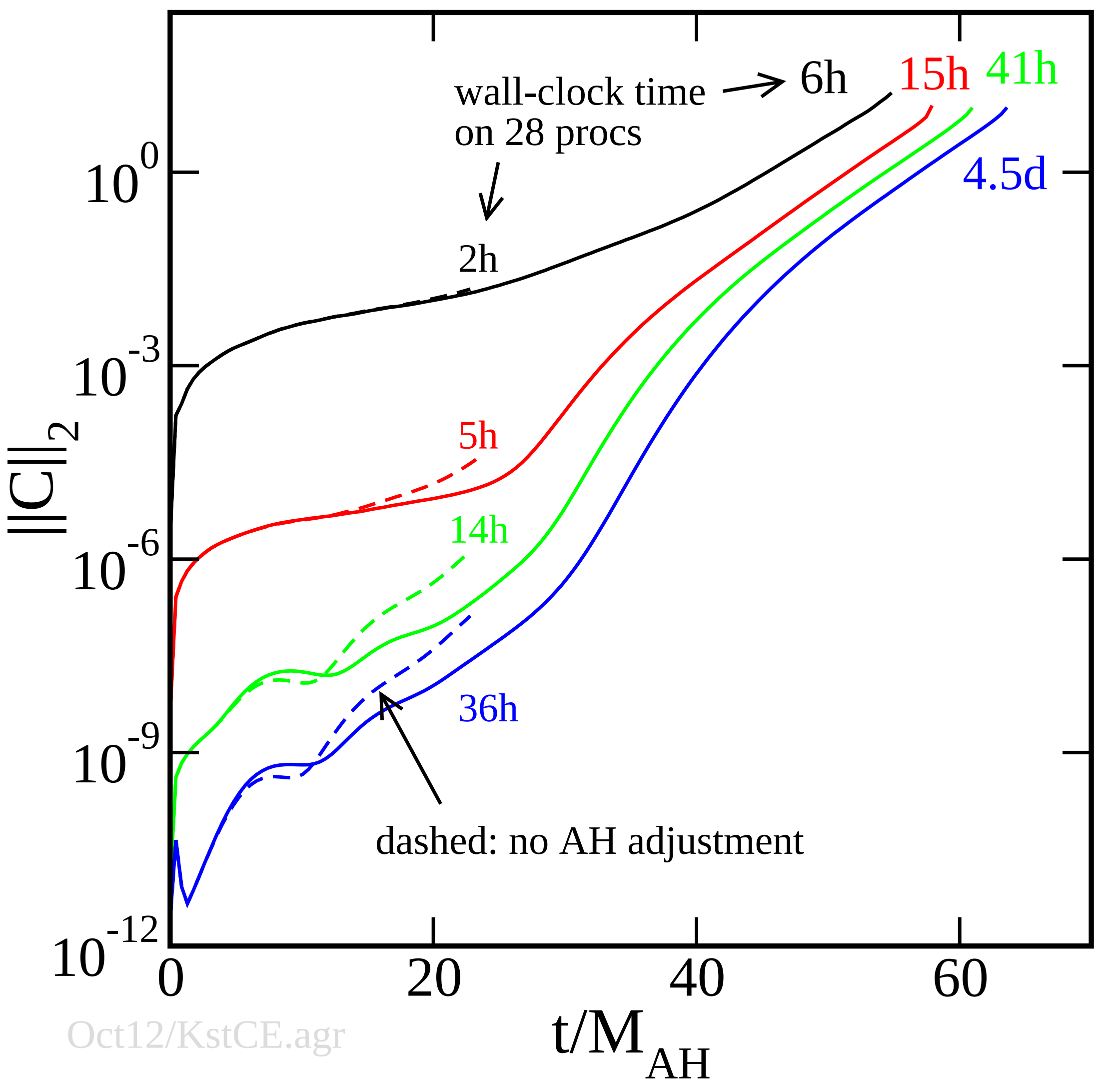

- Shift adjustment keeps horizons spherical.
  - But *ad hoc* shift inadequate to control coordinates everywhere.
- How to fix?
  - Driver gauge conditions.
  - Elliptic gauge conditions.

III. A New Generalized Harmonic Formulation

First-order generalized harmonic system

- Motivation:
  - Wish to reproduce Frans Pretorius' impressive generalized harmonic results.
  - Basic idea of generalized harmonic:

$\longrightarrow$ New gauge fields H_μ defined by $\Box x^\mu = \Gamma^{\mu\alpha}{}_\alpha \equiv H^\mu$
$\longrightarrow$ Only constraint is $\mathscr{C}_\mu \equiv H_\mu - \Gamma_{\mu\alpha}{}^\alpha$
$\longrightarrow$ Wave equations for 4-metric $g_{\mu\nu}$.

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  - Define variables as in Kashif Alvi's (2002) system:

$$\begin{aligned} g_{\alpha\beta} &\equiv 4-\mathrm{metric} \\ \Phi_{k\alpha\beta} &\equiv \partial_k g_{\alpha\beta} \\ \Pi_{\alpha\beta} &\equiv N^{-1}(\partial_t - N^k\partial_k) g_{\alpha\beta} \end{aligned}$$

  - $\Phi_{k\alpha\beta}$ variable introduces additional constraint: $\partial_k g_{\alpha\beta} - \Phi_{k\alpha\beta} \equiv \mathscr{C}_{kij} = 0.$

1st order generalized harmonic equations

- Multiples of constraints are added to evolution equations:

$$
\begin{aligned}
\partial_t g_{\alpha\beta} - (1+\gamma_1) N^k \partial_k g_{\alpha\beta} &= -N\Pi_{\alpha\beta} - \gamma_1 N^k \Phi_{k\alpha\beta} \\
\partial_t \Phi_{j\alpha\beta} &= \ldots + \gamma_2 N \mathcal{C}_{j\alpha\beta} \\
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  - γ_1 controls shocks: (linear degeneracy for $\gamma_1 = -1$).

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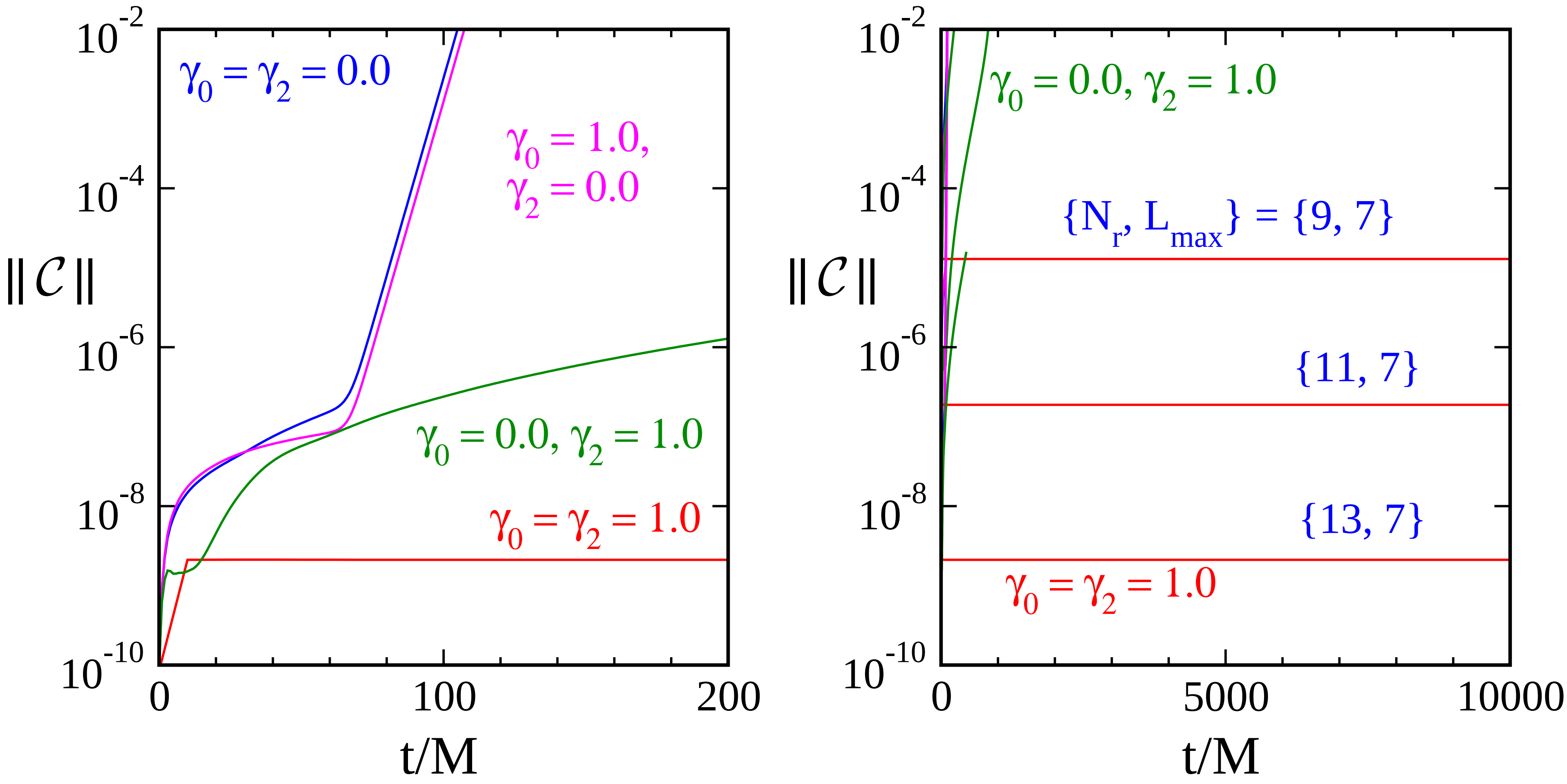
\partial_t g_{\alpha\beta} - (1+\gamma_1) N^k \partial_k g_{\alpha\beta} &= -N\Pi_{\alpha\beta} - \gamma_1 N^k \Phi_{k\alpha\beta} \\
\partial_t \Phi_{j\alpha\beta} &= \ldots + \gamma_2 N \mathscr{C}_{j\alpha\beta} \\
\partial_t \Pi_{\alpha\beta} &= \ldots + \gamma_0 N Q^{\mu}_{\alpha\beta} \mathscr{C}_{\mu} + \gamma_1 \gamma_2 N^k \mathscr{C}_{k\alpha\beta}
\end{aligned}
$$

  - γ_0 damps gauge constraint $\mathscr{C}_\mu$ (same damping term Frans uses).
  - γ_2 damps $\partial_k g_{\alpha\beta} - \Phi_{k\alpha\beta} \equiv \mathscr{C}_{k\alpha\beta}$ constraint.
  - γ_1 controls shocks: (linear degeneracy for $\gamma_1 = -1$).

- Evolution equations symmetric hyperbolic for **all** γ_0, γ_1 and γ_2.
- Constraint propagation equations symmetric hyperbolic for **all** γ_0, γ_1 and γ_2.

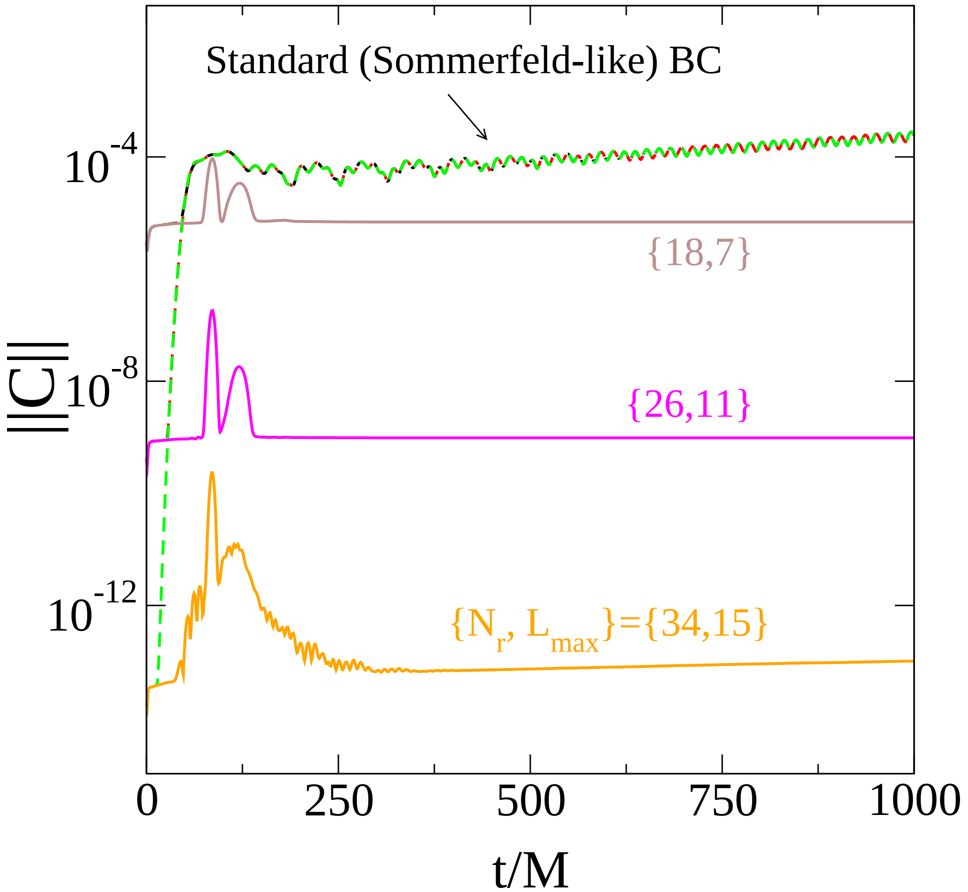
Constraint damping

- Single Schwarzschild BH in 3D.
  - Spherical shell domain, inner boundary $1.8M$, outer boundary $11.8M$
  - Standard (Sommerfeld-like) outer boundary conditions.

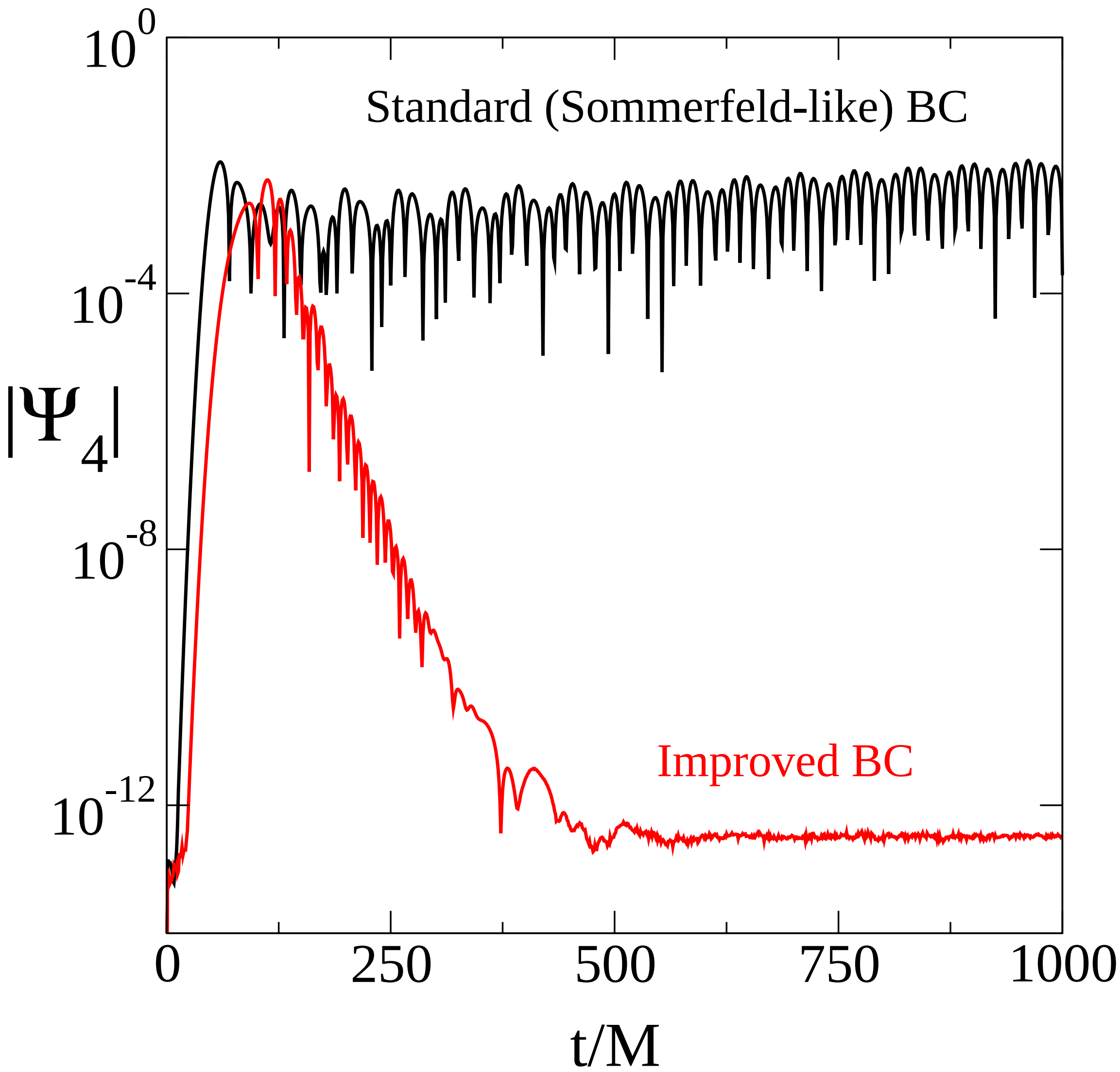


Improved boundary conditions

- We have constructed constraint-preserving and no-incoming-Weyl BCs.
  - Example: Schwarzschild with gravitational wave injected through boundary.
  - Domain = two concentric shells, outer boundary $23.6M$.

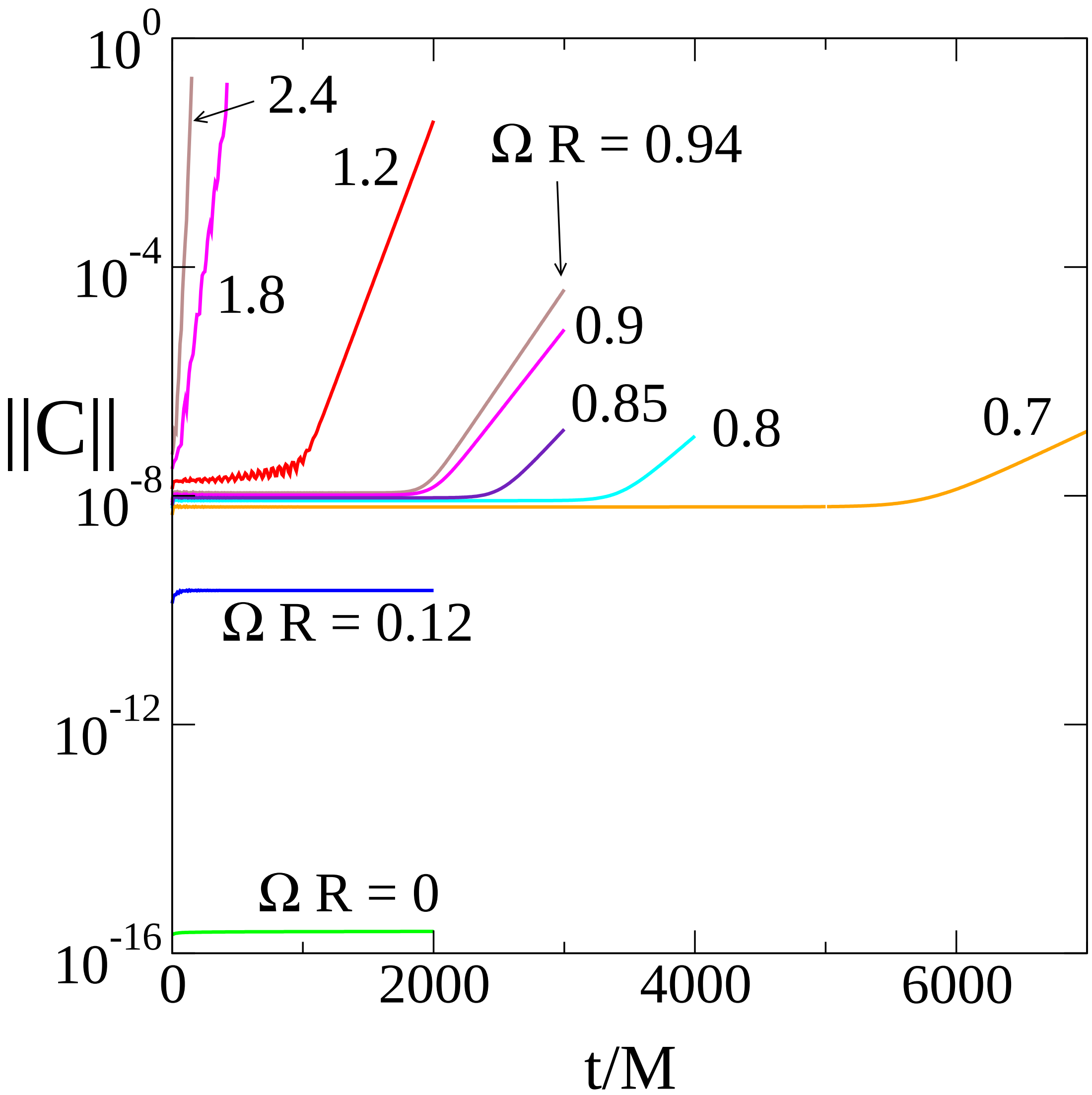


Improved boundary conditions



- $|\Psi_4|$ extracted at outer boundary $(r = 23.6M)$

Corotation problem?



- Flat space (!) in spherical shell of outer radius R.
- Coordinate system rotating at frequency Ω.
- KST system does not have this problem.

Summary

- Pseudospectral multidomain evolutions efficient.
- KST Binary evolution still has gauge and constraint problems. Future work:
  - Constraint projection.
  - Better gauge conditions (driver or elliptic).
- New 1st-order Generalized Harmonic promising.
  - Constraint damping parameters work well.
  - Constraint-preserving BCs effective.
  - Must solve corotation problem. (or move the holes?)

KST BBH blowup independent of domain decomposition