Predictions for the last stages of inspiral and plunge using analytical techniques

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Content:

- **How far we can push analytical calculations**

- **Original motivations of introducing resummation techniques which use post-Newtonian calculations**

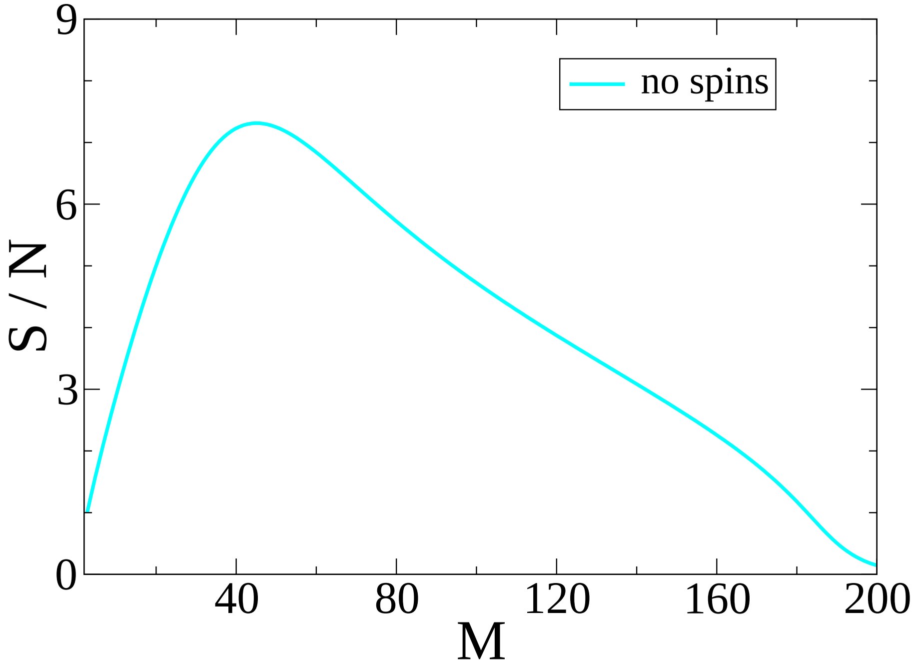
- **Transition from adiabatic inspiral to plunge for non-spinning and spinning, precessing binaries: main features of the dynamics and the waveforms**

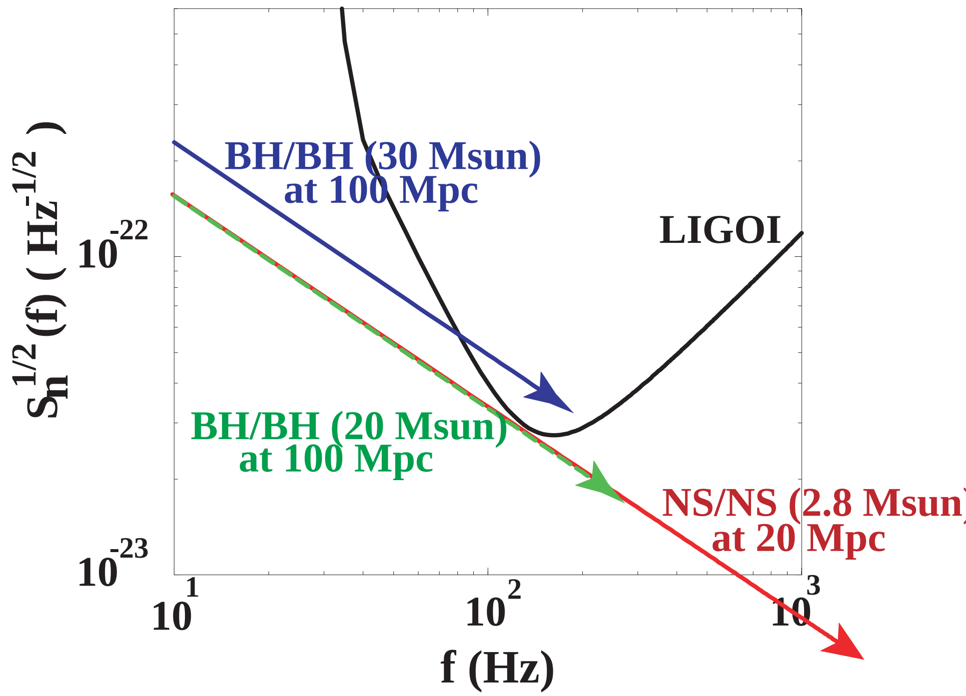
- **Comparison between analytical and numerical predictions**

- **What would be needed for a successful detection and for an accurate parameter estimation with ground- and space-based detectors**

Detectibility of inspiraling non-spinning binaries with LIGO-I

$$\mathcal{M} = M\nu^{3/5} \qquad M = m_1 + m_2 \qquad \nu = m_1 m_2/M^2 \qquad \frac{S}{N} \propto \frac{\mathcal{M}^{5/6}}{R}$$



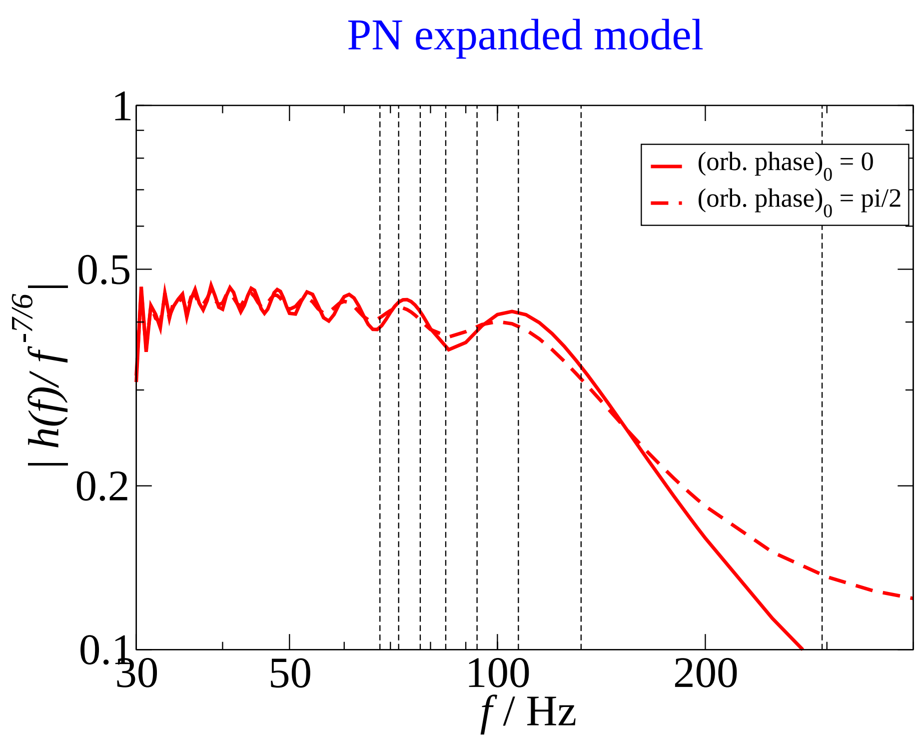


Equal-mass binaries at 100 Mpc

Reduction in signal power (with perfect match of the GW phase)

The significance of the last GW cycles

[AB, Chen & Vallisneri 02]



Reduction in signal power [continued]

[AB, Chen & Vallisneri 02]

	PN expanded model	
Number of cycles left	frequency	time-domain fractional sig. power
0	295.17	1.000
1	132.18	0.747
2	107.24	0.562
3	93.45	0.434
4	84.17	0.344
5	77.32	0.280
6	71.95	0.231
7	67.59	0.194

Table 1: Instantaneous frequencies and fractional signal power (SNR squared) when 0,1,2,. . . 7 GW cycles are left.

What determines the "adiabatic" waveforms

Inspiral: adiabatic sequence of circular orbits (quadrupole approximation)

$$h \propto v^2 \cos 2\varphi$$

Keplerian velocity: $v = (M\dot{\varphi})^{1/3}$ $\qquad M = m_1 + m_2$

$$\boxed{\text{Energy-balance equation: } \frac{dE(v)}{dt} = -F(v)}$$

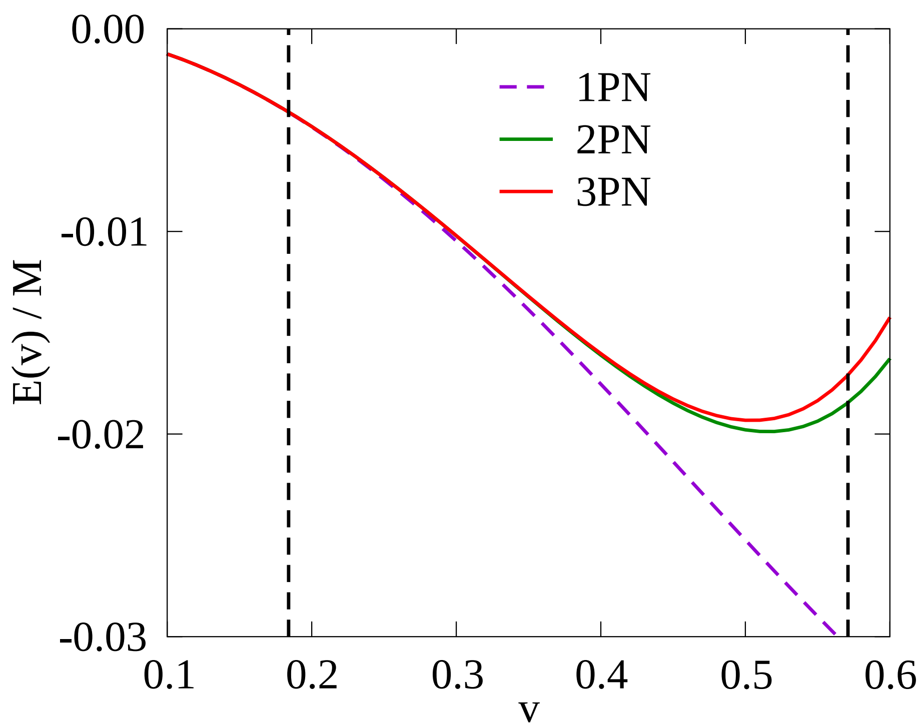
$E(v)$ and $F(v)$ known as a Post-Newtonian expansion in v/c

Two crucial ingredients:

$E(v) \longrightarrow$ center-of-mass energy $\qquad F(v) \longrightarrow$ gravitational flux

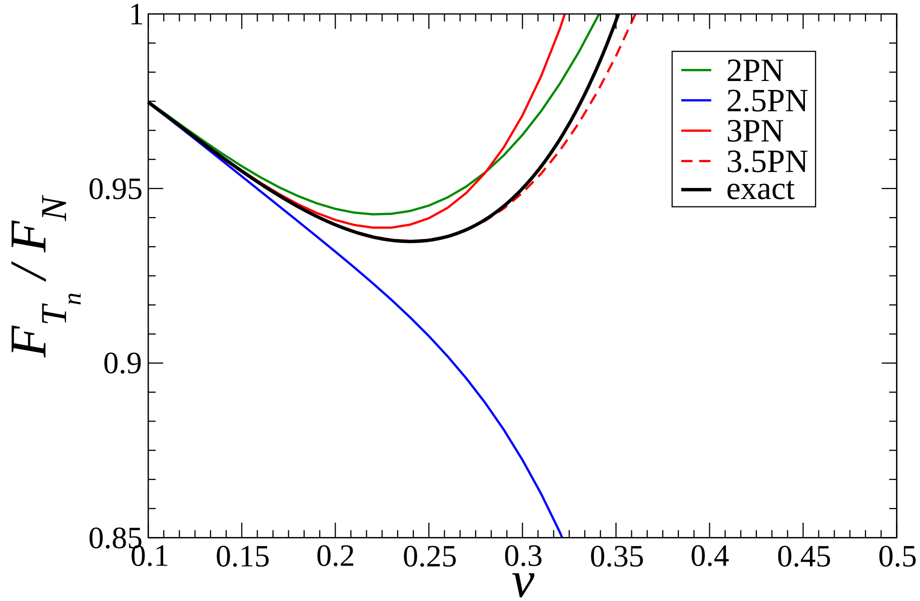
Initial motivations of introducing resummation techniques: PN-expanded circular-orbit energy

- **Circular-orbit energy determined at 2PN order in 1995 by** `Blanchet, Damour, Iyer Wiseman, Will` **and at 3PN order in 2001 by** `Damour, Jaranoswki & Schaefer`

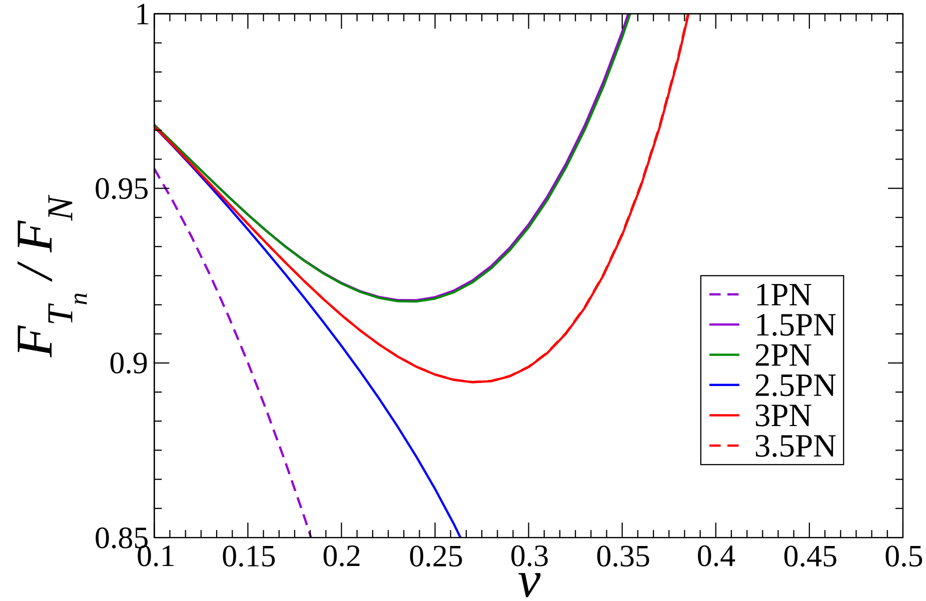


Initial motivations of introducing resummation techniques: PN-expanded GW flux

- **GW flux determined at 2.5PN order in 1996 by** `Blanchet` **and at 3PN and 3.5PN order in 2004 by** `Blanchet, Damour, Esposio-Farese & Iyer`

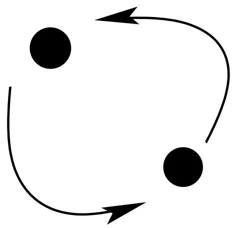


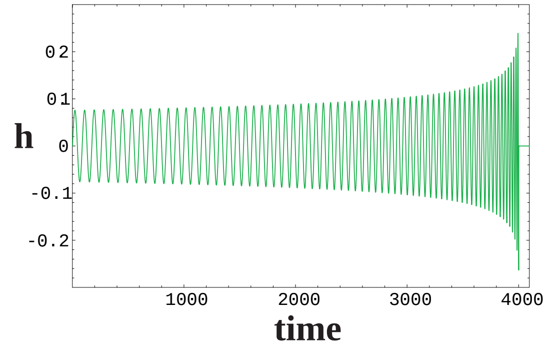
Test-mass limit case



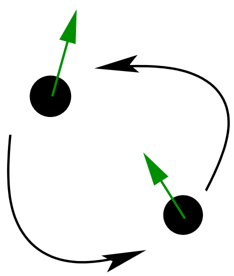
Equal-mass binaries

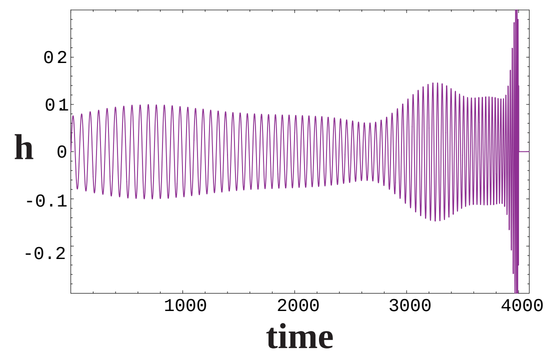
Precessing versus non-precessing compact binaries





- **Non spinning: Inspiral [$f_{\rm GW} = 2 f_{\rm orb}, f_{\rm end}(m_1, m_2)$], plunge, merger and ring down**





Many more parameters!

- **Precessing: Inspiral [$f_{\rm GW} = (2 f_{\rm orb}, f_{\rm prec})$, $f_{\rm end}(\boldsymbol{S}, m_1, m_2)$], plunge (?) merger, ring down**

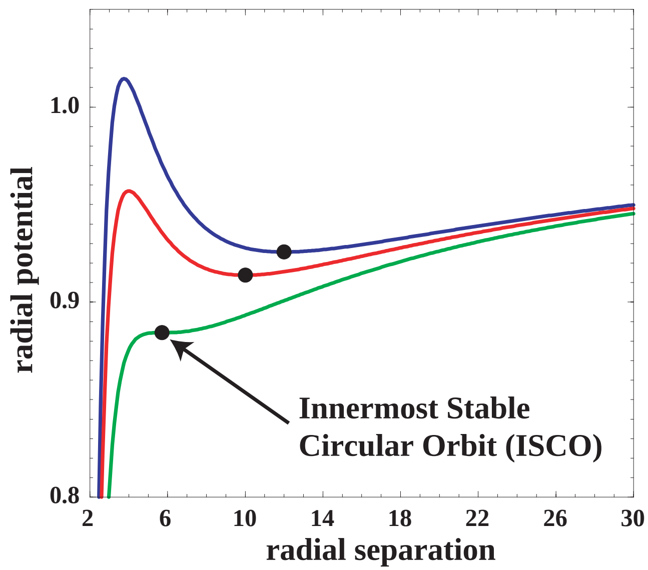
Precession of the orbital plane modulates both amplitude and phase of gravity-wave

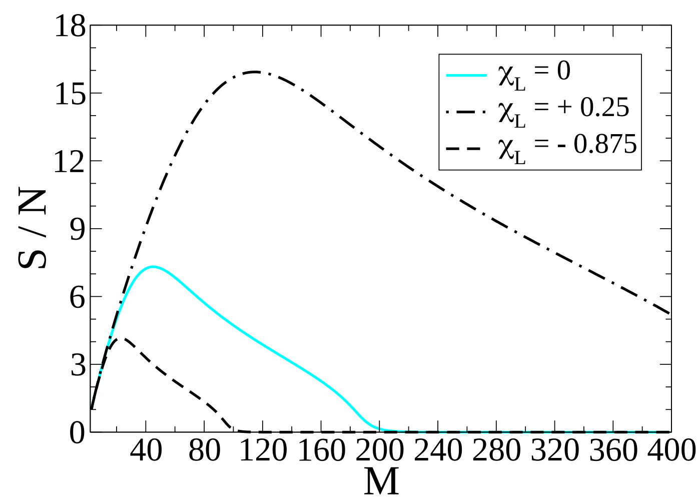
Detectibility of inspiraling spinning binaries with GW interferometers

Spin-orbit coupling makes two-body gravitational interaction more (less) repulsive when spins are aligned (anti-aligned)

$$V(r) = -\frac{mM}{r} + \frac{L^2}{2mr^2} - \frac{L^4}{m^3 r^4} + \cdots + \frac{2}{r^3}\boldsymbol{L}\cdot\boldsymbol{S} + \cdots$$

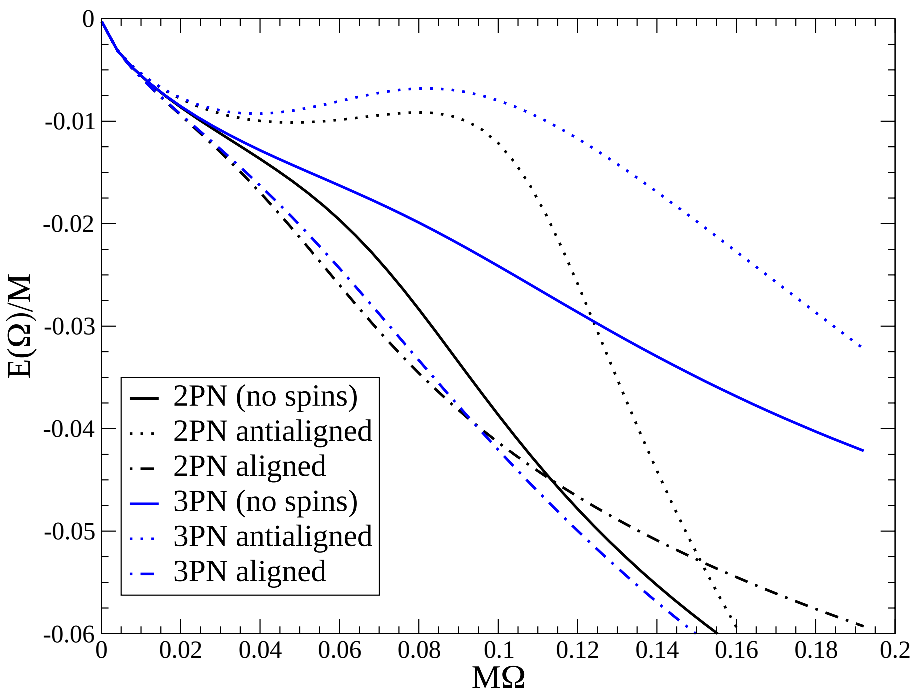
Duration of inspiral (and signal-to-noise ratio) modified by spin effects

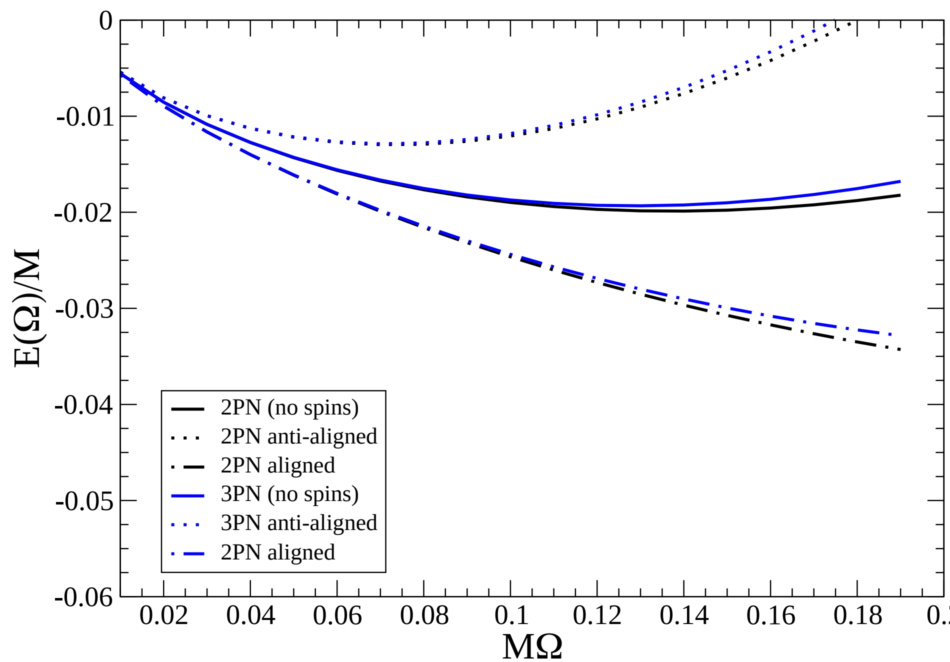




Equal-mass binaries at 100 Mpc

Circular-orbit energy for PN expanded models





Numerically evaluated: $\Omega = \frac{\partial H(r, p_r=0, p_\phi)}{\partial p_\phi}$

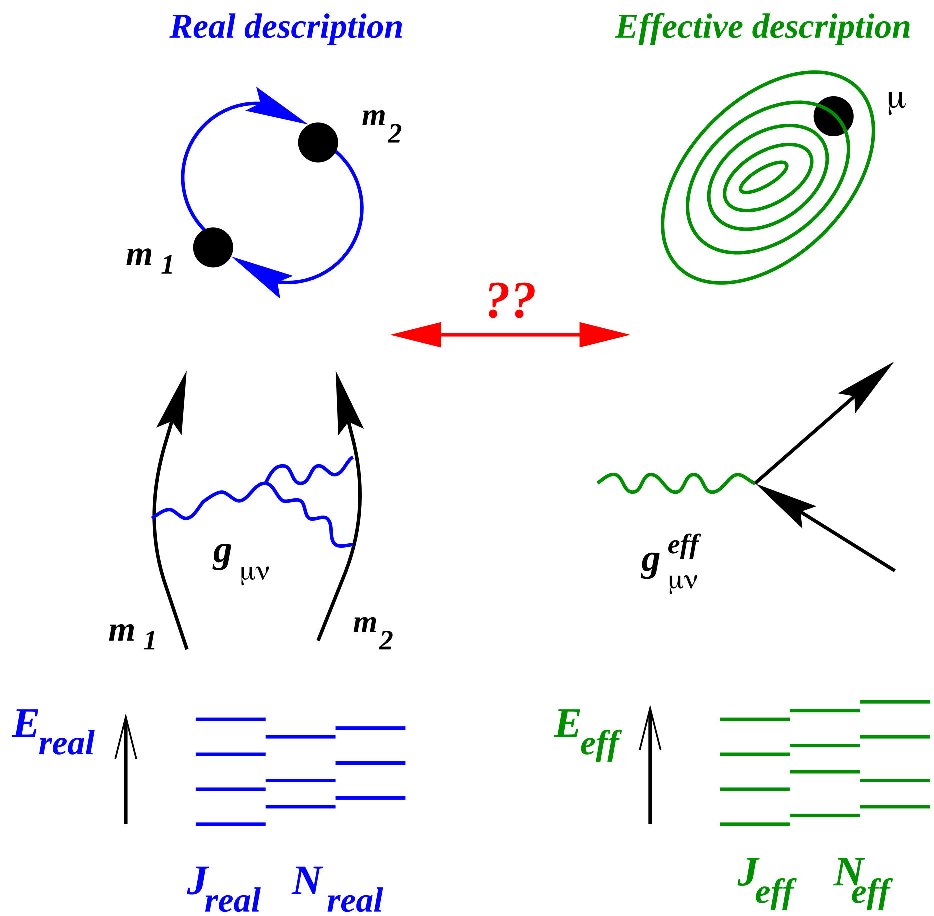
$$\frac{\partial H(r, p_r=0, p_\phi)}{\partial r} = 0 \quad \Rightarrow \quad p_\phi = p_\phi(r)$$

Analytically evaluated: keeping only terms until nPN order if working at nPN order

Effective-one-body approach

[AB & Damour 99]

$$\mu = m_1\, m_2/M$$
$$\nu = m_1\, m_2/M^2$$
$$0 \leq \nu \leq 1/4$$



EOB approach: resummed Hamiltonian (non-spinning black holes)

[AB & Damour 99]

"Effective" description

"Real" description

$$\mathcal{H}_{\rm real}(\boldsymbol{Q},\boldsymbol{P}) \sim M\left\{1+\nu\left[\frac{\boldsymbol{P}^2}{2}+\frac{M}{Q}\right]+c_4\,\boldsymbol{P}^4+\cdots\right\}$$

$$\mathcal{H}_{\rm eff}^{\nu}(\mathbf{q},\mathbf{p}) = \sqrt{A_\nu(q)\,\left[1+\mathbf{p}^2+\left(\frac{A_\nu(q)}{D_\nu(q)}-1\right)\,(\boldsymbol{n}\cdot\mathbf{p})^2+\mathcal{T}_4(\mathbf{p})\right]}$$

$$\boxed{\mathcal{H}_{\rm real}^{\rm improved}(\boldsymbol{Q},\boldsymbol{P}) = \sqrt{1+2\nu\,\left(\mathcal{H}_{\rm eff}^{\nu}(\mathbf{q},\mathbf{p})-1\right)}}$$

$$ds_{\rm eff}^2 = -A_\nu(q)\,dt^2+\frac{D_\nu(q)}{A_\nu(r)}\,dq^2+q^2\,d\Omega^2$$

- **Canonical transf. (resummed dynamics):** $\mathbf{q}=\mathcal{Q}(\boldsymbol{Q},\boldsymbol{P})$, $\mathbf{p}=\mathcal{P}(\boldsymbol{Q},\boldsymbol{P})$

- **All dynamics condensed in** $A_\nu(q)$ **and** $D_\nu(q)$**!**

New resummed *orbital energy* **function:** $E_{\rm real}^{\rm impr}(v)$

Effective one-body approach at 3PN

[Damour, Jaranowski & Schafer 00]

At 3PN order: one more equation to satisfy than number of unknowns

Higher order derivatives in the effective description
$$\boxed{0 = m_0^2 + g_{\rm eff}^{\alpha\beta}\, p_\alpha\, p_\beta + A^{\alpha\beta\gamma\delta}\, p_\alpha\, p_\beta\, p_\gamma\, p_\delta + \cdots}$$

Same matching between real and effective energy

$$\mathcal{H}^{\nu}_{\rm eff,3PN}(\mathbf{q},\mathbf{p}) = \sqrt{A_\nu(q)\ \left[\cdots + z_1\,\frac{\mathbf{p}^4}{q^2} + z_2\,\frac{\mathbf{p}^2\,(\boldsymbol{n}\cdot\mathbf{p})^2}{q^2} + z_3\,\frac{(\boldsymbol{n}\cdot\mathbf{p})^4}{q^2}\right]}$$

Result for effective metric at 2PN order and beyond it

$$ds^2_{\rm eff} = -A_\nu(q)\, c^2\, dt^2 + \frac{D_\nu(q)}{A_\nu(q)}\, dq^2 + q^2\, d\Omega^2$$

$$A_\nu(q) = 1 - 2\,\frac{GM}{c^2 q} + 2\nu\, \left(\frac{GM}{c^2 q}\right)^3 \qquad D_\nu(q) = 1 - 6\nu\, \left(\frac{GM}{c^2 q}\right)^2$$

Effective potential: $W_j(q) = A_\nu(q)\, [1 + \frac{j^2}{q^2}]$

Location of the ISCO: $\frac{\partial W_j}{\partial q} = 0 = \frac{\partial^2 W_j}{\partial q^2}$

At higher PN orders: $A_\nu(q) = 1 - 2\,\frac{GM}{c^2 q} + 2\nu\, \left(\frac{GM}{c^2 q}\right)^3 + 18.7\nu\, \left(\frac{GM}{c^2 q}\right)^4 + \mathcal{O}\left(\frac{GM}{c^2 q}\right)^5$

Possible resummation of $A_\nu(q)$ to improve its behaviour

e.g., Padé approximants [Damour, Jaranowski & Schaefer 00]

EOB approach with spins

- **Approximate map of the conservative dynamics of two spinning black holes of mass m_1 and m_2 onto the dynamics of a non-spinning particle of mass $\mu = m_1\, m_2/M$ moving in an effective metric** [Damour 01]

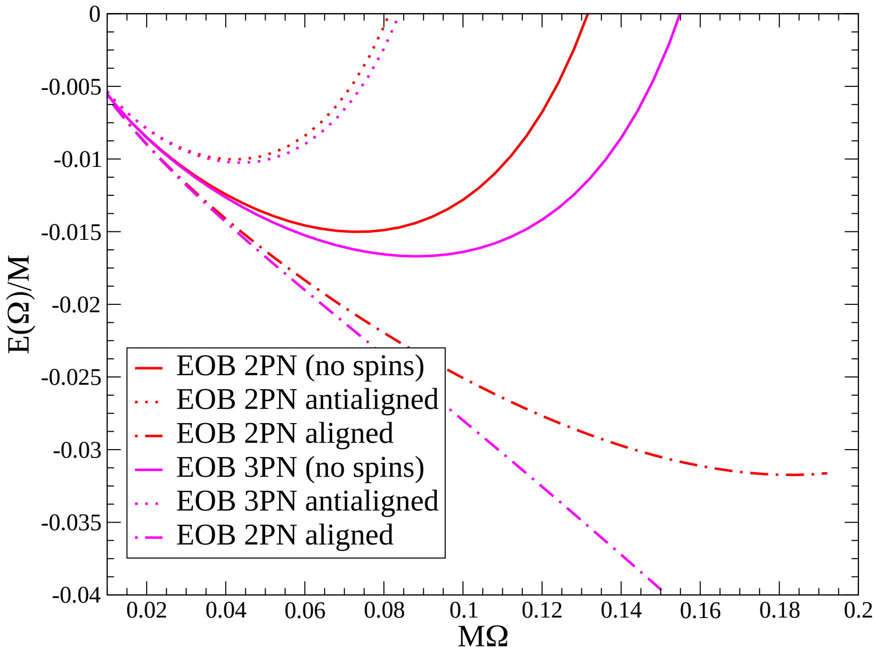
- **This metric can be viewed as a $\nu = \mu/M$ deformation of a Kerr metric of mass $M = m_1 + m_2$ and spin $S_{\rm eff}$**

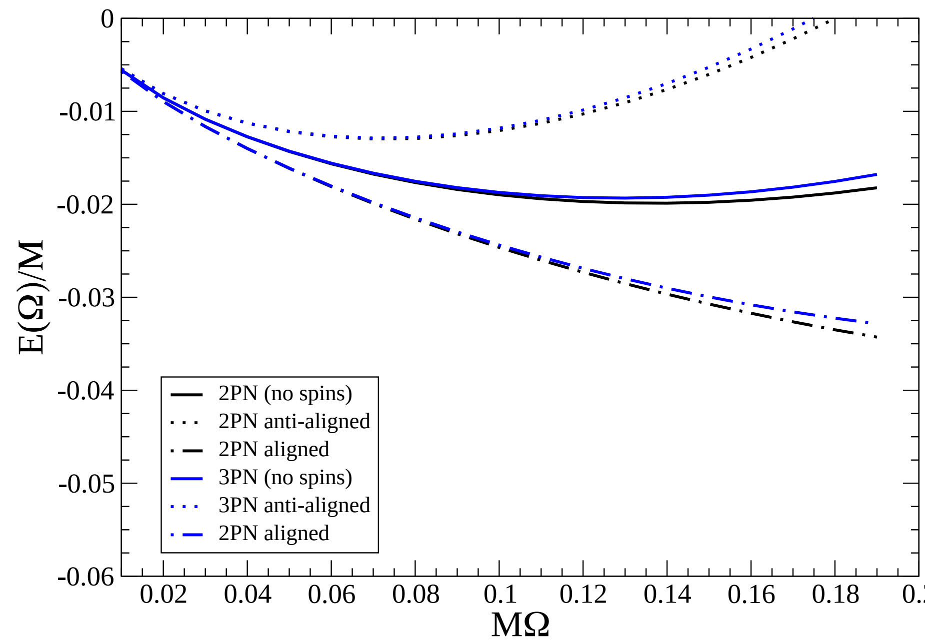
For simplicitly, we just added spin effects to the non-spinning EOB Hamiltonian [AB, Chen & Damour 05]

$$\mathcal{H}^{\rm impr}_{\rm real}(\mathbf{q},\mathbf{p},\boldsymbol{S}_1,\boldsymbol{S}_2) = \mathcal{H}^{\rm impr}_{\rm real}(\mathbf{q},\mathbf{p}) + H_{\rm SO}(\mathbf{q},\mathbf{p},\boldsymbol{S}_1,\boldsymbol{S}_2) + \mathcal{H}_{\rm SS}(\mathbf{q},\mathbf{p},\boldsymbol{S}_1,\boldsymbol{S}_2)$$

$$\mathcal{H}_{SO} = \frac{2\boldsymbol{S}_{\rm eff}\cdot\boldsymbol{L}}{q^3}\,, \quad \boldsymbol{S}_{\rm eff} \equiv \left(1 + \frac{3}{4}\frac{m_2}{m_1}\right)\boldsymbol{S}_1 + \left(1 + \frac{3}{4}\frac{m_1}{m_2}\right)\boldsymbol{S}_2$$

Comparing EOB-resummed and PN-expanded binding energies for equal-mass binaries



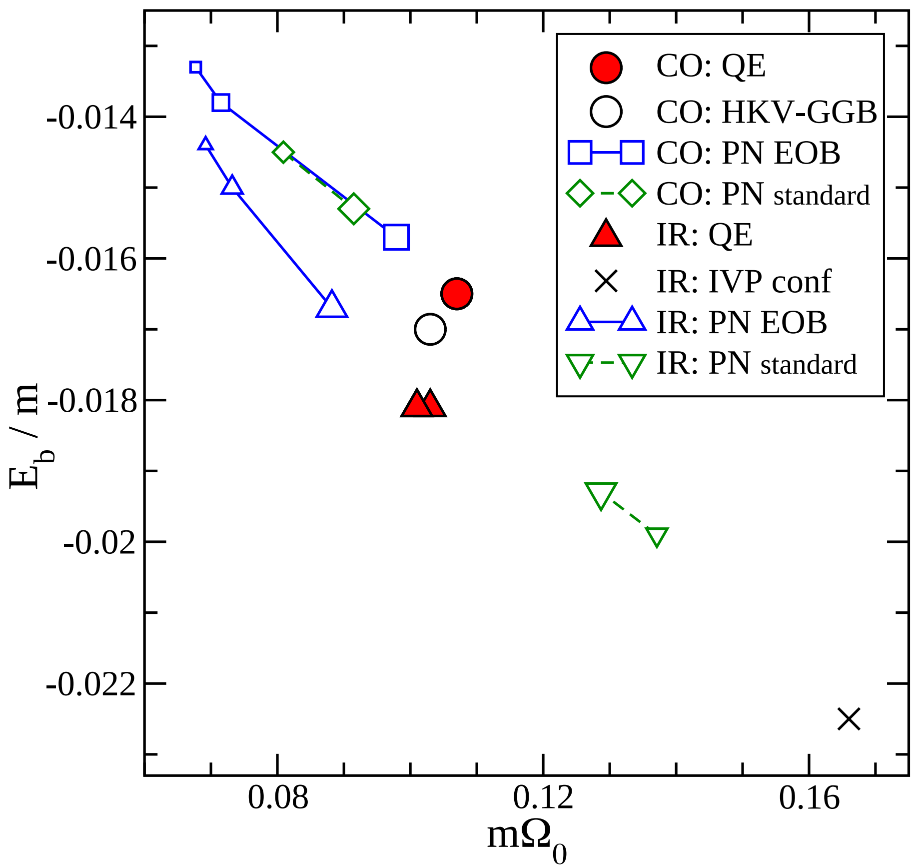


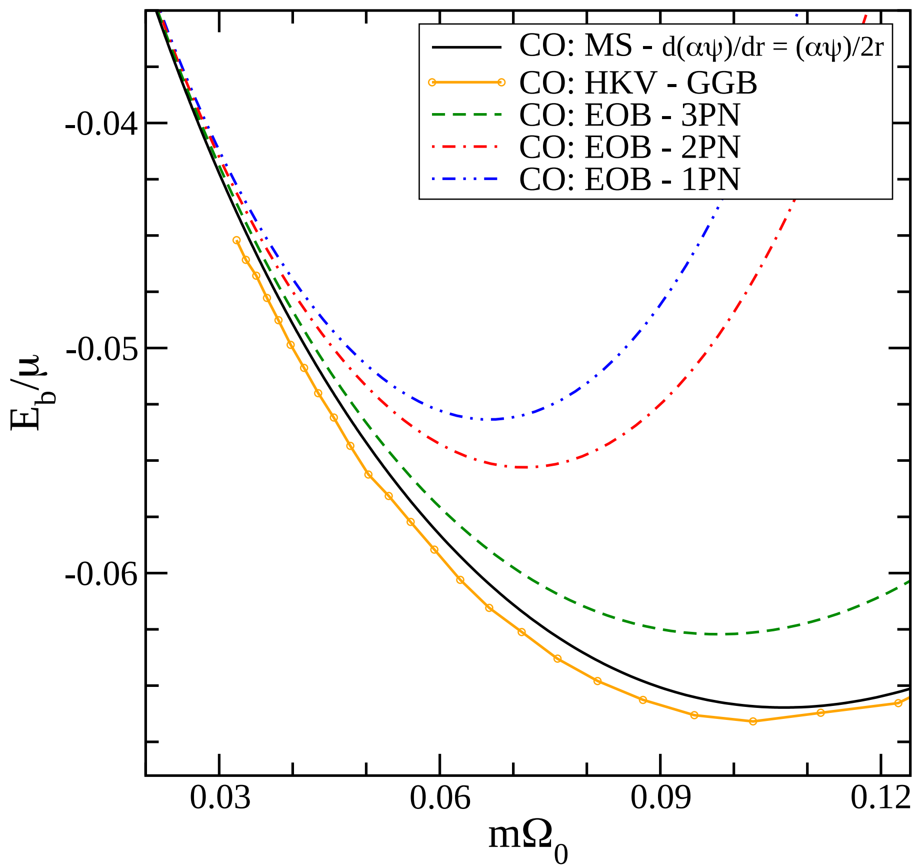
Numerically evaluated: $\Omega = \frac{\partial \mathcal{H}^{\rm impr}(q,p_q{=}0,p_\phi)}{\partial p_\phi}$

$$\frac{\partial \mathcal{H}^{\rm impr}(q,p_q{=}0,p_\phi)}{\partial q} = 0 \quad \Rightarrow \quad p_\phi = p_\phi(q)$$

Analytically evaluated: keeping only terms until nPN order if working at nPN order

Comparing analytical and numerical results



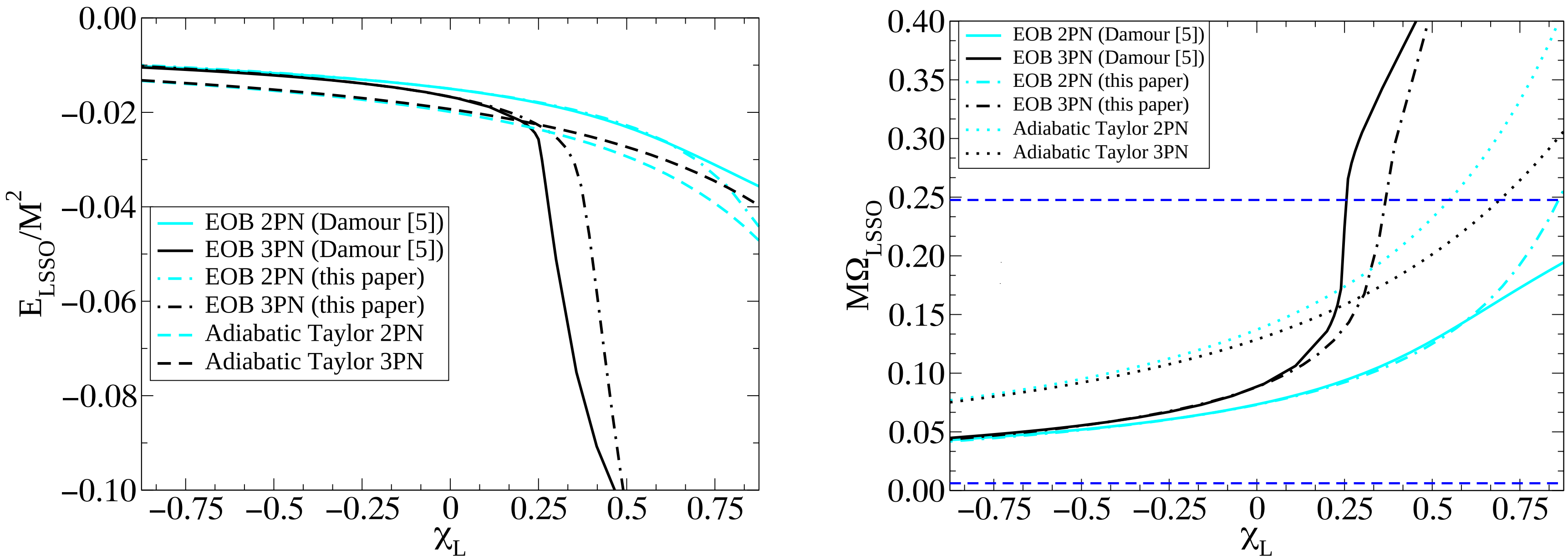


[Damour, Gourgoulhon & Grandéclement 02; Cook & Pfeiffer 04]

Comparing LSSO predictions for energy and frequency

Equal-mass and equal-spin binaries

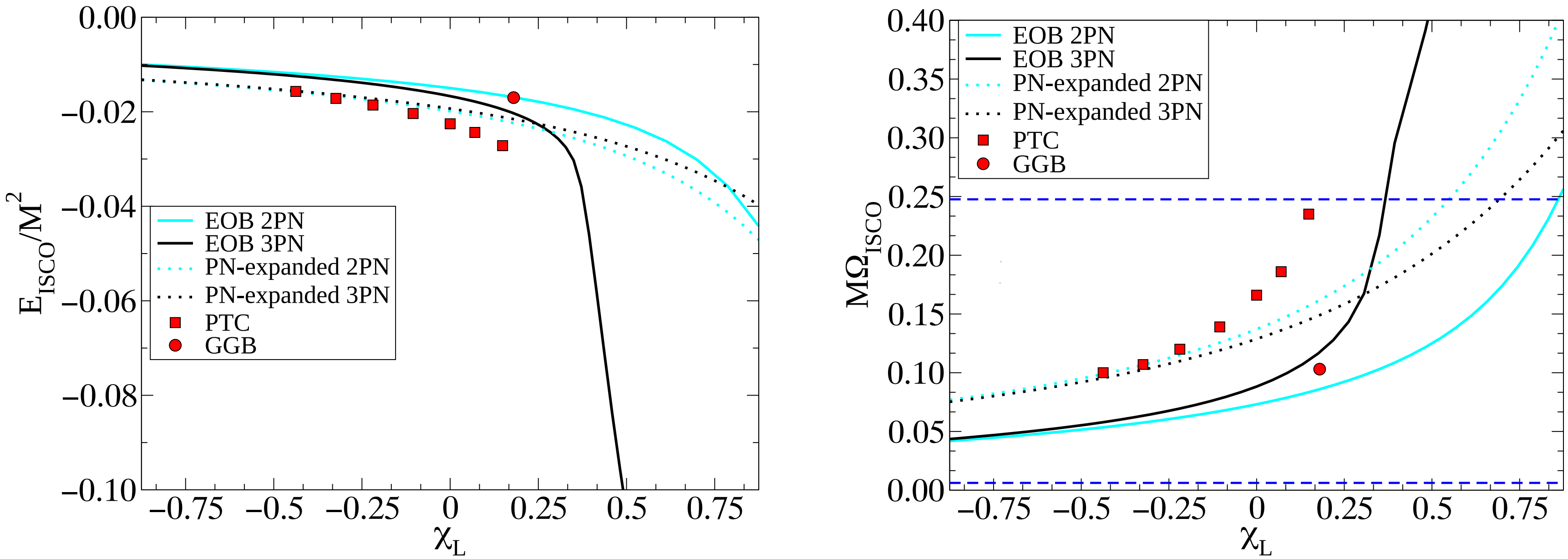
[AB, Chen & Damour 05]



For spins aligned with angular momentum $\Rightarrow$ non-linear effects dominate $\Rightarrow$ predictions differ, but for LIGO this would affect *only* **binaries with mass $\gtrsim 40 M_\odot$**

Comparing LSSO predictions using analytical calculations and *old* results from initial-value problem approach

Equal-mass and equal-spin binaries [AB, Chen & Damour 05]



- **"Effective potential"', Initial-value-problem approach** [Pfeiffer, Teukolsky & Cook 00]
- **HKV-, QE-approach** [Grandéclement, Gourgoulhon, Bonazzola 02; Cook 02; Cook & Pfeiffer 04]

EOB approach: incorporating radiation reaction effects

[AB & Damour 00; AB, Chen & Damour 05]

$$\frac{dq^i}{dt} = \frac{\partial \mathcal{H}^{\rm impr}}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}^{\rm impr}}{\partial q^i} + \mathcal{F}_i$$

- **Assumptions: quasi-circular orbits and leading spin-dependent terms**

- **Radiation-reaction force matches known rates of energy and angular momentum loss for quasi-adiabatic orbits**

$$\mathcal{F}_i = \frac{1}{\Omega\,|\boldsymbol{L}|}\frac{dE}{dt}\,p_i + \frac{8}{15}\,\nu^2\,\frac{v^8}{\boldsymbol{L}^2{}_q}\left\{\left(61 + 48\,\frac{m_2}{m_1}\right)\mathbf{p}\cdot\boldsymbol{S}_1 + \left(61 + 48\,\frac{m_1}{m_2}\right)\mathbf{p}\cdot\boldsymbol{S}_2\right\}\,L_i$$

- **Padé resummation of the GW flux including spin effects**

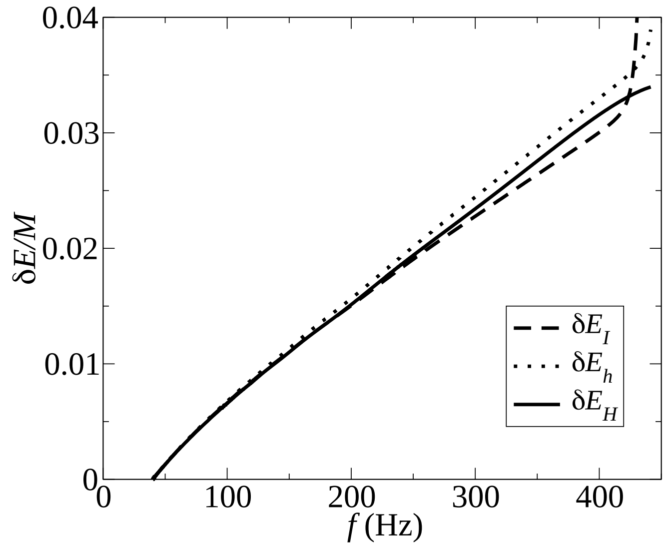
[Damour, Sathyaprakash & Iyer 98; Porter & Sathyaprakash 04; AB, Chen & Damour 05]

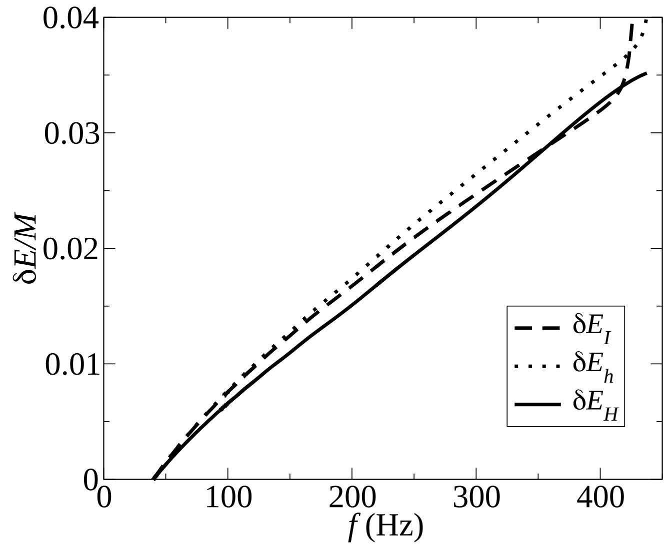
Evaluation of waveform and energy released

Quadrupole approximation: [AB & Damour 00; AB, Chen & Damour 05]

$$h_{ij} = \frac{H_{ij}}{D} \equiv \frac{2\mu}{D}\frac{d^2}{dt^2}(q_i\,q_j), \quad \ddot{q}_k = -M\,q_k/q^3 \Rightarrow H_{ij} = 4\mu\left(V_i\,V_j - M\frac{q_i\,q_j}{q^3}\right)$$

- $\delta\mathcal{H}$



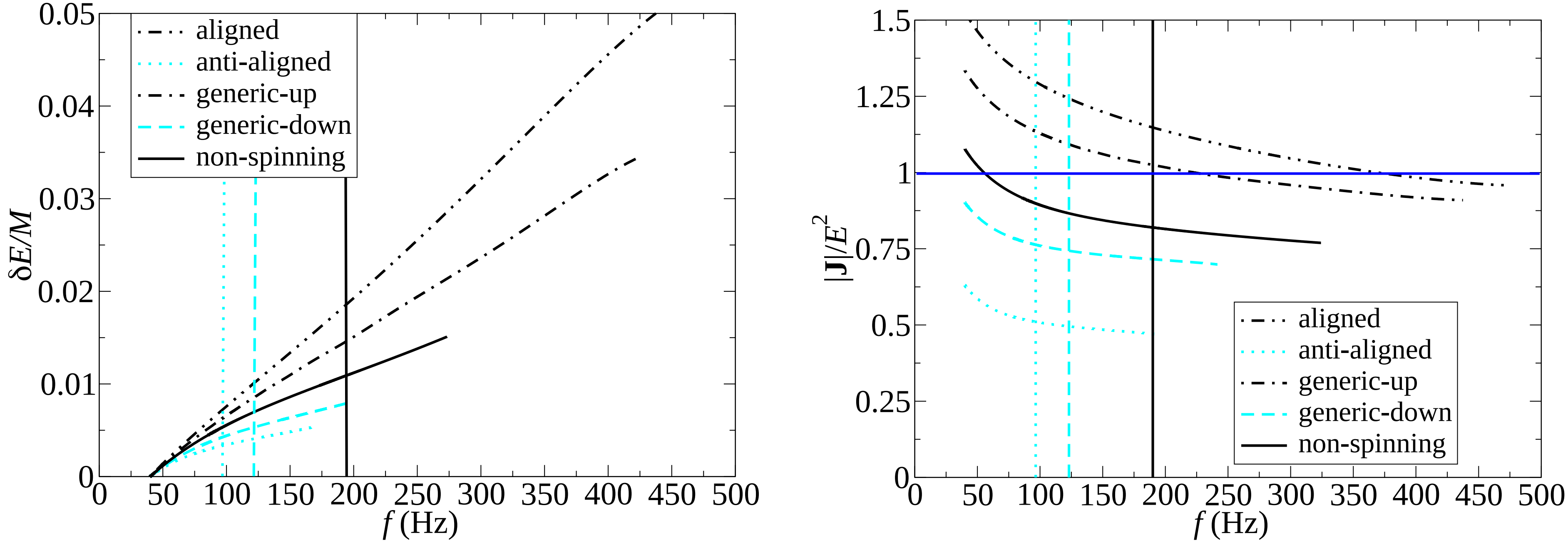


- The time integral of E_h with $\frac{dE_h}{dt} = \frac{1}{20}\int \sum_{ij} \dot{H}_{ij}^{\mathrm{TF}}\,\dot{H}_{ij}^{\mathrm{TF}}$
- The time integral of E_I with $\frac{dE_I}{dt} = \frac{1}{5}\frac{d^3 I_{ij}}{dt^3}\frac{d^3 I_{ij}}{dt^3}$ $I_{ij} = \mu\,\left(q_i\,q_j - \frac{1}{3}\delta_{ij}\,q^k\,q_k\right)$

Energy and angular-momentum released during inspiral and plunge

[AB, Chen & Damour 05]

- **Maximal spins and** $(15+15)M_\odot$
- **Energy release before** $40\,\mathrm{Hz}$ **is** $\sim 0.008/M$



Rotation parameter J/E^2 smaller than one at the end of inspiral

$\Rightarrow$ **Kerr black hole could already form**

Energy and angular-momentum released until 40 Hz and from 40 Hz to the LSSO

$(\theta_{S1}, \phi_{S1}, \theta_{S2}, \phi_{S2})$	$[\delta E_H]_{f<40\,\mathrm{Hz}}/M$	f_{LSSO} (Hz)	$[\delta E_H]^{40\,\mathrm{Hz}}_{\mathrm{LSSO}}/M$	$\left[|\mathbf{J}|/E^2\right]_{\mathrm{LSSO}}$
		$(15+15)M_\odot$, 3PN		
nospin	0.0082	190	0.0107	0.82
$(0^\circ,0^\circ,0^\circ,0^\circ)$	0.0086	(1430)	—	—
$(180^\circ,0^\circ,180^\circ,0^\circ)$	0.0077	97	0.0033	0.51
$(60^\circ,90^\circ,60^\circ,0^\circ)$	0.0084	(767)	—	—
$(120^\circ,90^\circ,120^\circ,0^\circ)$	0.0079	123	0.0054	0.74
		$(15+5)M_\odot$, 3PN		
nospin	0.0048	265	0.0084	0.62
$(0^\circ,0^\circ,0^\circ,0^\circ)$	0.0049	(1442)	—	—
$(180^\circ,0^\circ,180^\circ,0^\circ)$	0.0046	140	0.0034	0.14
$(60^\circ,90^\circ,60^\circ,0^\circ)$	0.0049	(798)	—	—
$(120^\circ,90^\circ,120^\circ,0^\circ)$	0.0047	177	0.0049	0.62

Energy and angular-momentum released until 40 Hz and from 40 Hz up to the end of evolution

$(\theta_{S1}, \phi_{S1}, \theta_{S2}, \phi_{S2})$	$[\delta E_H]_{f<40\,\mathrm{Hz}}/M$	f_{fin}	$[\delta E_H]^{40,\mathrm{Hz}}_{\mathrm{fin}}/M$	$\left[\lvert\mathbf{J}\rvert/E^2\right]_{\mathrm{fin}}$
$(15+15)M_\odot$, 3PN				
nospin	0.0082	325	0.0183	0.77
$(0^\circ,0^\circ,0^\circ,0^\circ)$	0.0086	474	0.0528	0.96
$(180^\circ,0^\circ,180^\circ,0^\circ)$	0.0077	194	0.0064	0.47
$(60^\circ,90^\circ,60^\circ,0^\circ)$	0.0084	440	0.0353	0.91
$(120^\circ,90^\circ,120^\circ,0^\circ)$	0.0079	242	0.0101	0.70
$(15+5)M_\odot$, 3PN				
nospin	0.0048	484	0.0141	0.58
$(0^\circ,0^\circ,0^\circ,0^\circ)$	0.0049	817	0.0495	0.95
$(180^\circ,0^\circ,180^\circ,0^\circ)$	0.0046	289	0.0054	0.11
$(60^\circ,90^\circ,60^\circ,0^\circ)$	0.0049	706	0.0292	0.91
$(120^\circ,90^\circ,120^\circ,0^\circ)$	0.0047	354	0.0080	0.60

No a priori obstacles at having a Kerr black hole form right after the end of the non-adiabatic “plunge” $\Rightarrow$ no ground for expecting a large emission of GWs between plunge and merger

Energy released from the LSSO up to the end of the evolution

$(\theta_{S1}, \phi_{S1}, \theta_{S2}, \phi_{S2})$	$f_{\rm LSSO}$ (Hz)	$f_{\rm fin}$	$[\delta E_H]_{\rm fin}^{\rm LSSO}/M$
$(15+15)M_\odot$, 3PN			
nospin	190	325	0.0075
$(0^\circ,0^\circ,0^\circ,0^\circ)$	1430	474	0.0527
$(180^\circ,0^\circ,180^\circ,0^\circ)$	97	194	0.0031
$(60^\circ,90^\circ,60^\circ,0^\circ)$	760	440	0.0353
$(120^\circ,90^\circ,120^\circ,0^\circ)$	123	242	0.0047
$(15+5)M_\odot$, 3PN			
nospin	265	484	0.0057
$(0^\circ,0^\circ,0^\circ,0^\circ)$	1442	819	0.0493
$(180^\circ,0^\circ,180^\circ,0^\circ)$	140	289	0.0024
$(60^\circ,90^\circ,60^\circ,0^\circ)$	793	719	0.0294
$(120^\circ,90^\circ,120^\circ,0^\circ)$	177	351	0.0031

Energy released from the LSSO up to the end of the evolution [continued]

Comparison with Flanagan & Hughes 97; Baker, Bruegmann, Campanelli, Lousto & Takahashi 00; Baker, Campanelli, Lousto & Takahashi 04:

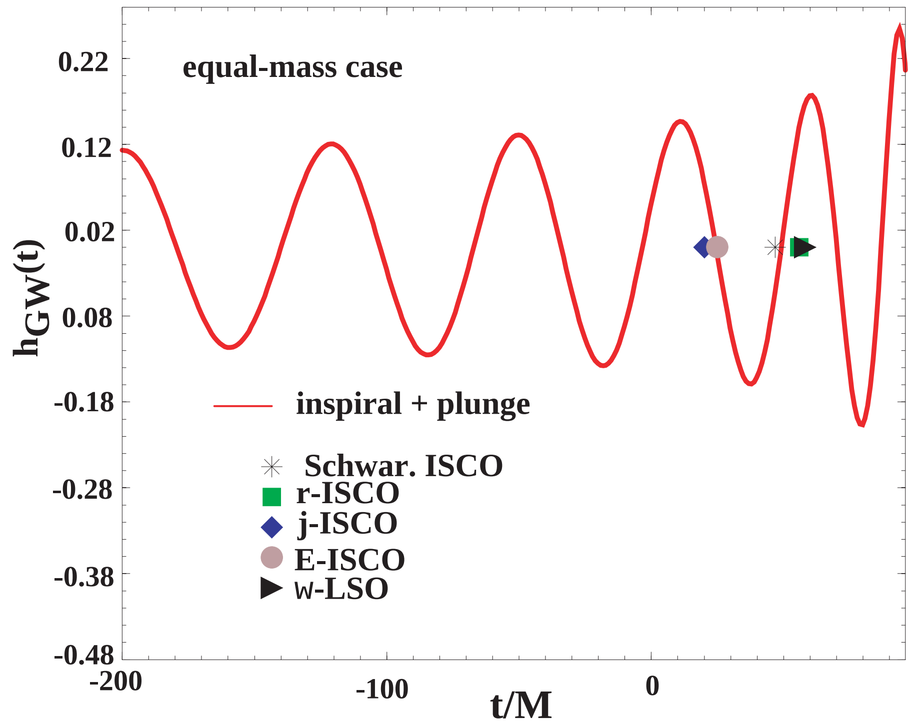
- **No spin:** 1.4% **of** M **against** $3 - 4\%$ **of** M **by BBCLT**

- **Small spins: differences of few percent with BCLT but they include also ring-down**

However BBCL and BCLT use IVP formulation for initial data

- **With spins: energy released not as large as predicted by a rough estimate of** Flanagan & Hughes

Comparable-mass case: several possible definitions of ISCO crossing



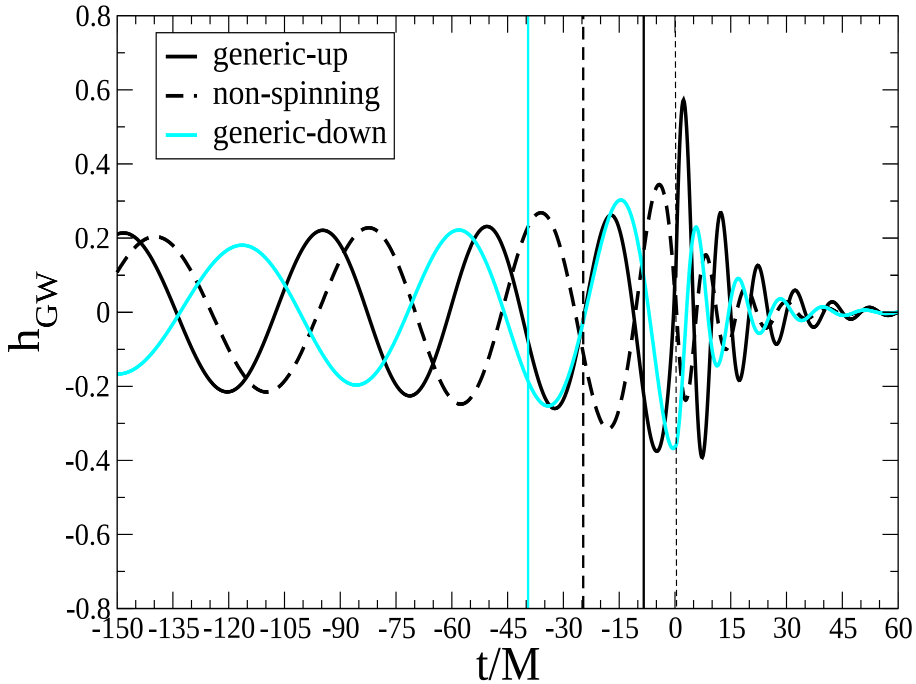
Radiation reaction effects rather large $\Rightarrow$ transition to the plunge blurred

Gravity-wave signal from inspiral-plunge(–ring-down)

[AB, Chen & Damour 05]

- $\chi_1 = \chi_2 = 0.5$
  and $(15 + 15)M_\odot$

- $M_{\rm BH} = E_{\rm fin}$
  and $a_{\rm BH} = [\boldsymbol{J}/E^2]_{\rm fin}$



- **When spins are present**
  the ring-down part is just an example: we restricted to $l = m = 2$, assuming that the total angular momentum is dominated by the orbital angular momentum

Summary

- **Within analytical calculations the EOB is the only approach which can describe the dynamics and the gravity-wave signal beyond the adiabatic approximation**

- **It can provide initial data $(\mathrm{q}, \mathrm{p}, g_{ij}, k_{ij})$ for black holes close to the plunge to be used by numerical relativity**

- **It can be used as a diagnostic for (or to fit) numerical relativity results**

- **Current results indicate good agreement between numerical and analytical estimate of the binding energy without spin effects. Predictions (using, e.g., HKV and QE methods) which include spin couplings are needed**

Summary [continued]

- **Waveforms generated from initial data compatible with analytical calculations are needed**

- **Extension of EOB to NS-NS and NS-BH** [AB, Damour & Gourgoulhon]

- **Detection with LIGO/VIRGO: phenomenological templates or extensions of EOB templates can cover possible differences between analytical and numerical waveforms for the last stages of inspiral and plunge**

- **Accurate parameter estimation and tests of GR with LIGO and LISA: we would need more accurate waveforms for late inspiral and plunge**

- **Only the detection (and coalescence waves from NR!) will reveal us if the two-body problem is a smooth deformation of a one-body problem, at least from the point of view of the gravitational-wave emission**

Where the waveforms from NR will be?

[AB, Chen & Vallisneri 02]